统计力学

CQI 博士生资格考试复习小组

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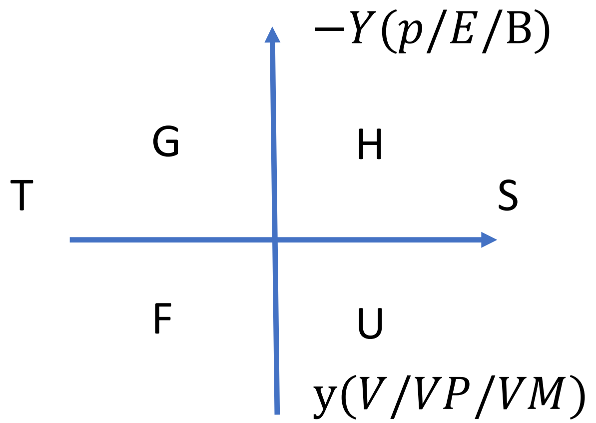
热力学 Cheatsheet

- 气体的三个方便实验测量的常数α β κ_T
- 常用微分关系：循环关系、互逆关系、链式关系、脚标关系
- 物态方程：理想气体/范德瓦尔斯气体/昂尼斯方程/简单固液/顺磁性物质
- 功的表达形式：体积功/薄膜表面/电介质/磁介质
- 微分关系与麦氏关系
- 热容的定义
- 相变潜热公式与克拉伯龙公式



概览

- 知识点总结（参考资料 汪志诚《热力学·统计物理》）
  - 第六章 近独立粒子的最概然分布
  - 第七章 玻尔兹曼统计
  - 第八章 波色统计和费米统计
  - 第九章 系综理论
- 历年题目

知识点总结：近独立粒子的最概然分布

- 粒子运动状态的描述 --> 经典描述和量子描述

$$\varepsilon = \varepsilon(q_1, \dots, q_r; p_1, \dots, p_r)$$

- 经典描述：自由粒子 线性谐振子 转子

$$\varepsilon = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

$$\varepsilon = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$\varepsilon = \frac{1}{2I}\left(P_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2\right) \rightarrow \varepsilon = \frac{p_\varphi^2}{2I}$$

知识点总结：近独立粒子的最概然分布

- 粒子运动状态的描述 --> 经典描述和量子描述

$\varepsilon = \varepsilon(q_1, \dots, q_r; p_1, \dots, p_r)$

- 量子描述：线性谐振子 转子 自旋角动量 自由粒子

$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right)$

$\varepsilon = \frac{L^2}{2I}; L^2 = l(l+1)\hbar^2 (l = 0,1,2,\cdots); L_z = m\hbar(m = -l, -l+1, \cdots, l)$

$\varepsilon = -\vec{\mu} \cdot \vec{B} = \frac{e\hbar m_s}{m} B_z$

$\varepsilon = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right); p_i = \frac{h}{L} n_i; dn_x dn_y dn_z = \frac{V}{h^3} dp_x\, dp_y\, dp_z$

知识点总结：系统微观运动状态的描述

- 全同：由具有完全相同的内禀属性的同类粒子组成
- 近独立：粒子之间的相互作用很弱
- 等概率原理：对于处在平衡状态的孤立系统，系统各个可能的微观状态出现的概率是相等的
- 分布
  - 能级 $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l, \cdots$
  - 简并度 $\omega_1, \omega_2, \cdots, \omega_l, \cdots$
  - 粒子数 $a_1, a_2, \cdots, a_l, \cdots$

知识点总结：系统微观运动状态的描述

- 玻尔兹曼系统 $\Omega_{M.B.} = \frac{N!}{\prod_l a_l!} \prod_l \omega_l^{a_l}$ （n全排列为$n!$）
- 波色系统 $\Omega_{B.E.} = \prod_l \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$ （使用$\omega_l - 1$个隔板）
- 费米系统 $\Omega_{F.D.} = \prod_l \frac{\omega_l!}{a_l!(\omega_l - a_l)!}$ （ω_l里面选前a_l个）
- 当$\frac{a_l}{\omega_l} \ll 1$时，$\Omega_{B.E.} \approx \frac{\Omega_{M.B.}}{N!}$， $\Omega_{F.D.} \approx \frac{\Omega_{M.B.}}{N!}$ （经典极限条件）

知识点总结：M\B\F分布

- 玻尔兹曼系统 $\Omega_{M.B.} = \frac{N!}{\prod_l a_l!} \prod_l \omega_l^{a_l}$
- 由 $\delta \ln \Omega - \alpha \delta N - \beta \delta E = 0 \Rightarrow a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}$
- 同理
  - 玻尔兹曼分布 $a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}$
  - 玻色分布 $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$
  - 费米分布 $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$
- 关系：非简并条件下，$e^{\alpha} \gg 1$，遵循玻尔兹曼分布
  - 弱简并 → 一阶近似

知识点总结：玻尔兹曼统计

- $a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}$

积分公式：$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

- 配分函数 $Z_1 = \sum_l \omega_l e^{-\beta\varepsilon_l} = \frac{1}{h^3}\int e^{-\beta\varepsilon(p,q)}dp\,dq$
- $N = e^{-\alpha} Z_1 \quad U = -N\frac{\partial}{\partial\beta}\ln Z_1$
- $Y = -\frac{N}{\beta}\frac{\partial}{\partial y}\ln Z_1 \rightarrow p = \frac{N}{\beta}\frac{\partial}{\partial V}\ln Z_1, M = \frac{n}{\beta}\frac{\partial}{\partial B}\ln Z_1, \ldots$
- $S = Nk\left(\ln Z_1 - \beta\frac{\partial}{\partial\beta}\ln Z_1\right) = k\ln\Omega$
  - $S = Nk\left(\ln Z_1 - \beta\frac{\partial}{\partial\beta}\ln Z_1\right) - k\ln N! = k\ln\frac{\Omega}{N!}$（经典极限）
- $F = U - TS = -NkT\ln Z_1, \quad \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$

知识点总结：玻尔兹曼统计

- 标准解题程序：利用配分函数（举个栗子 同核双原子分子）

$$\varepsilon=\frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)+\left(n+\frac{1}{2}\right)\hbar\omega+\frac{l(l+1)\hbar^2}{2I}$$

$Z_1 \rightarrow C_V \cdots$

注意：振动的简并度为1，转动的简并度为2l+1，转动在高温下变求和为积分求解

- 标准解题程序：利用配分函数（举个栗子 异核双原子分子）

$$\varepsilon=\frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)+\frac{1}{2m_\mu}\left(p_r^2+m_\mu\omega^2r^2\right)+\frac{1}{2I}\left(L_\theta^2+\frac{1}{\sin^2\theta}L_\varphi^2\right)$$

$Z_1 \rightarrow C_V \cdots$

知识点总结：玻尔兹曼统计

- 应用：
  - 理想气体物态方程
  - 麦克斯韦速度分布→最概然速率v_s<平均速率$\bar{v}$ <方均根速率v_m
  - 碰壁数 $\Gamma = \frac{1}{4} n\bar{v}$
  - 能量均分定理：在M.B.分布下，$\overline{\frac{1}{2}ap^2} = \frac{1}{2}kT$，$\overline{\frac{1}{2}bq^2} = \frac{1}{2}kT$
  - 通过能量均分定理，简化瑞利金斯公式的推导
    - $U_\omega d\omega = \frac{V}{\pi^2 c^3}\omega^2 kT d\omega$（用到$k_i = \frac{2\pi}{L} n_i$和$\omega = ck$）
  - 固体热容的爱因斯坦理论
    - $\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right) \to Z_1 \to U \to C_V \to$ 高温/低温
  - 顺磁性固体 $\varepsilon = \pm \boldsymbol{\mu} \cdot \boldsymbol{B}$

知识点总结：波色/费米统计

- $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \mp 1}$
- 配分函数 $\Xi = \prod_l \Xi_l = \prod_l \left(1 \mp e^{-\alpha-\beta\varepsilon_l}\right)^{-\omega_l}$
- $\alpha = -\frac{\mu}{kT} \quad \beta = \frac{1}{kT}$
- $\bar{N} = -\frac{\partial}{\partial\alpha}\ln\Xi \quad U = -\frac{\partial}{\partial\beta}\ln\Xi$
- $Y = -\frac{1}{\beta}\frac{\partial}{\partial y}\ln\Xi$ → $p = \frac{1}{\beta}\frac{\partial}{\partial V}\ln\Xi$
- $S = k\left(\ln\Xi + \alpha\bar{N} + \beta U\right) = k\ln\Omega$

知识点总结：波色/费米统计

- 应用：
  - 波色体系 波色-爱因斯坦凝聚
    - 1) $a_l > 0$ 2) $n_0 + \int a_l D(\varepsilon) d\varepsilon = \sum a_l = n$
  - 波色体系 光子气体
    - 两种方法：统计的方法$U = \sum_l \varepsilon_l a_l$；巨配分函数
    - 出发点：$\varepsilon = \hbar\omega = cp$；光子数不守恒$\Rightarrow \alpha = 0$，$\mu = 0$
  - 费米体系 金属中的自由电子气
    - 两种方法：统计的方法$N = \sum_l a_l$；巨配分函数
- 技巧：
  - $\sum_l \omega_l \to \int D(\varepsilon) d\varepsilon = \frac{V}{h^3} \iiint dp_x dp_y dp_z$
    - $\sum_l \omega_l \to \int D(\varepsilon) d\varepsilon = \frac{V}{2\pi^2 c^3} \int_0^\infty \omega^2 \, d\omega \times 2$
    - $\sum_l \omega_l \to \int D(\varepsilon) d\varepsilon = \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \int_0^\infty \sqrt{\varepsilon} \, d\varepsilon$

知识点总结：系综理论

- 从近独立粒子到互作用粒子
- 刘维尔定理：随着一个代表点沿正则方程所确定的轨道在相空间中运动，其邻域的代表点密度是不随时间改变的常数

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\sum_{\alpha}\left(\frac{\partial\rho}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}-\frac{\partial\rho}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right)=0$$

- 分布函数
  - 含义 $P=\rho(q,p,t)d\Omega$
  - 归一化 $\int\rho(q,p,t)d\Omega=1$
  - 宏观量 $\bar{B}(t)=\int B(q,p)\rho(q,p,t)d\Omega$

知识点总结：微正则系综

- 孤立系统 确定的粒子数N、体积V、能量E
- 等概率原理
  - $\rho(q,p) = \begin{cases} 1/\Omega & E \le H(q,p) \le E + \Delta E \\ 0 & H(q,p) < E \text{ or } E + \Delta E < H(q,p) \end{cases}$
- 直接的微正则系综计算较难，但是它是分析其他系综的基础

知识点总结：微正则系综

- 解题套路：$\Omega \rightarrow S = k\ln\Omega \rightarrow U \rightarrow p, T$
- 例题：
  - 设理想单原子气体有N个，$E = \sum_{i=1}^{3N} \frac{p_i^2}{2m}$，试求系统对应的$\Omega(E)$，并求出其它热力学量

知识点总结：正则系综

- 确定的粒子数N、体积V、温度T
- 推导思路
  - 热源远大于系综 & 等概率原理 $\rho_s \propto \Omega_r\left(E^{(0)} - E_s\right)$
  - $\ln \Omega_r\left(E^{(0)} - E_s\right) = \ln \Omega_r\left(E^{(0)}\right) - \beta E_s \Rightarrow \rho_s = \frac{1}{Z} e^{-\beta E_s}$
  - 状态变为能级 $Z = \sum_l \Omega_l e^{-\beta E_l}$ $\rho_l = \frac{1}{Z} \Omega_l e^{-\beta E_l}$
- $U = -N \frac{\partial}{\partial \beta} \ln Z, Y = -\frac{N}{\beta} \frac{\partial}{\partial y} \ln Z, S = Nk \left(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z\right)$
- 涨落 $\overline{(E - \bar{E})^2} = -\frac{\partial \bar{E}}{\partial \beta} = kT^2 \frac{\partial \bar{E}}{\partial T} = kT^2 C_V$

知识点总结：正则系综

- 解题套路： $E \to \mathrm{Z} \to$ 力学量
- 应用：
  - 实际气体的物态方程（计及分子间相互作用）
    - $E=\sum_{i=1}^{N}\frac{p_i^2}{2m}+\sum_{i<j}\phi(r_{ij}) \to Z \to f_{ij}=e^{-\beta\phi(r_{ij})}-1 \to$ 第二位力系数 $B \to$ 范德瓦尔斯方程
  - 固体的热容
    - $E=\phi_0+\sum_{i=1}^{3N}\hbar\omega_i\left(n_i+\frac{1}{2}\right)$ & $(\omega=ck \to D(\omega)d\omega) \to U \to C_V$

知识点总结：巨正则系综

- 确定的化学势μ、体积V、温度T
- 推导思路
  - 热源远大于系综 & 等概率原理 $\rho_s \propto \Omega_r\left(N^{(0)}-N_s, E^{(0)}-E_s\right)$
  - $\ln\Omega_r\left(N^{(0)}-N_s, E^{(0)}-E_s\right)=\ln\Omega_r\left(N^{(0)}, E^{(0)}\right)-\alpha N-\beta E_s \quad \Rightarrow \rho_s=\frac{1}{\Xi}e^{-\alpha N-\beta E_s}$
  - 状态变为能级 $\Xi=\sum_{N=0}^{\infty}\sum_s e^{-\alpha N-\beta E_l}=\sum_N \frac{e^{-\alpha N}}{N!h^{Nr}}\int e^{-\beta E(q,p)}d\Omega$
- $\bar{N}=-\frac{\partial}{\partial\alpha}\ln\Xi$ $U=-\frac{\partial}{\partial\beta}\ln\Xi$ $Y=-\frac{1}{\beta}\frac{\partial}{\partial y}\ln\Xi$ $S=k\left(\ln\Xi+\alpha\bar{N}+\beta U\right)$
- 涨落 $\overline{(N-\bar{N})^2}=-\left(\frac{\partial\bar{N}}{\partial\alpha}\right)|_{T,V}=kT^2\left(\frac{\partial\bar{E}}{\partial T}\right)|_{T,V}=\frac{kT}{V}\kappa_T$

例题

I. BOLTZMANN DISTRIBUTION

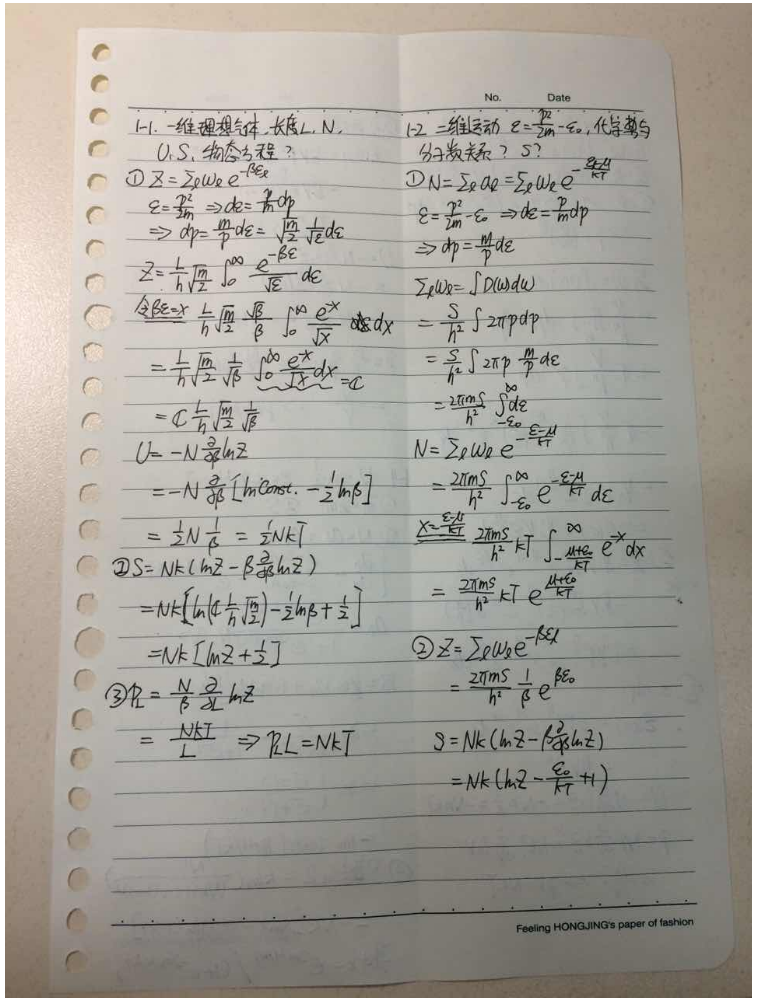
1. 一维长度为L的理想气体，分子数为N，求系统的内能，熵，状态方程。（07秋）

2. 分子在固体吸附面上做二维运动$\epsilon = \frac{p^2}{2m} - \epsilon_0$，求被吸附分子的化学势与吸附面上平均分子数的关系，并求体系的熵。（05春）

3. 考虑由能谱关系为$\epsilon = \alpha p^s$（α为常数，s=1,2）的粒子组成的n维经典理想气体。(a)试证明粒子的能态密度$D(\epsilon) = BV\epsilon^{\frac{n}{s}-1}$，B为常数。(b)求粒子的配分函数。(c)求气体的内能和物态方程。（03秋）

4. 晶格中N个自旋1/2的粒子处在均匀磁场H中，能量可以为$\pm m_0 H$，m_0是粒子的磁矩。温度为T。（1）求系统的总磁矩。（2）求系统的熵。（07春）

解答



No. Date

1-1. 一维理想气体，长度 L，N，U，S，物态方程？

① $Z = \sum_l \omega_l e^{-\beta\varepsilon_l}$

$\varepsilon = \frac{p^2}{2m} \Rightarrow d\varepsilon = \frac{p}{m}dp$

$\Rightarrow dp = \frac{m}{p}d\varepsilon = \sqrt{\frac{m}{2}}\frac{1}{\sqrt{\varepsilon}}d\varepsilon$

$Z = \frac{L}{h}\sqrt{\frac{m}{2}}\int_0^\infty \frac{e^{-\beta\varepsilon}}{\sqrt{\varepsilon}}d\varepsilon$

$\underline{令\beta\varepsilon = x}\ \frac{L}{h}\sqrt{\frac{m}{2}}\frac{\sqrt{\beta}}{\beta}\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$

$= \frac{L}{h}\sqrt{\frac{m}{2}}\frac{1}{\sqrt{\beta}}\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx = C$

$= C\frac{L}{h}\sqrt{\frac{m}{2}}\frac{1}{\sqrt{\beta}}$

$U = -N\frac{\partial}{\partial\beta}\ln Z$

$= -N\frac{\partial}{\partial\beta}[\ln Const. - \frac{1}{2}\ln\beta]$

$= \frac{1}{2}N\frac{1}{\beta} = \frac{1}{2}NkT$

② $S = Nk(\ln Z - \beta\frac{\partial}{\partial\beta}\ln Z)$

$= Nk[\ln(C\frac{L}{h}\sqrt{\frac{m}{2}}) - \frac{1}{2}\ln\beta + \frac{1}{2}]$

$= Nk[\ln Z + \frac{1}{2}]$

③ $P_L = \frac{N}{\beta}\frac{\partial}{\partial L}\ln Z$

$= \frac{NkT}{L} \Rightarrow P_L L = NkT$

1-2 二维运动 $\varepsilon = \frac{p^2}{2m} - \varepsilon_0$，化学势与分子数关系？S？

① $N = \sum_l a_l = \sum_l \omega_l e^{-\frac{\varepsilon-\mu}{kT}}$

$\varepsilon = \frac{p^2}{2m} - \varepsilon_0 \Rightarrow d\varepsilon = \frac{p}{m}dp$

$\Rightarrow dp = \frac{m}{p}d\varepsilon$

$\sum_l \omega_l = \int D(\omega)d\omega$

$= \frac{S}{h^2}\int 2\pi p\,dp$

$= \frac{S}{h^2}\int 2\pi p\frac{m}{p}d\varepsilon$

$= \frac{2\pi mS}{h^2}\int_{-\varepsilon_0}^\infty d\varepsilon$

$N = \sum_l \omega_l e^{-\frac{\varepsilon-\mu}{kT}}$

$= \frac{2\pi mS}{h^2}\int_{-\varepsilon_0}^\infty e^{-\frac{\varepsilon-\mu}{kT}}d\varepsilon$

$\underline{x = \frac{\varepsilon-\mu}{kT}}\ \frac{2\pi mS}{h^2}kT\int_{-\frac{\mu+\varepsilon_0}{kT}}^\infty e^{-x}dx$

$= \frac{2\pi mS}{h^2}kT e^{\frac{\mu+\varepsilon_0}{kT}}$

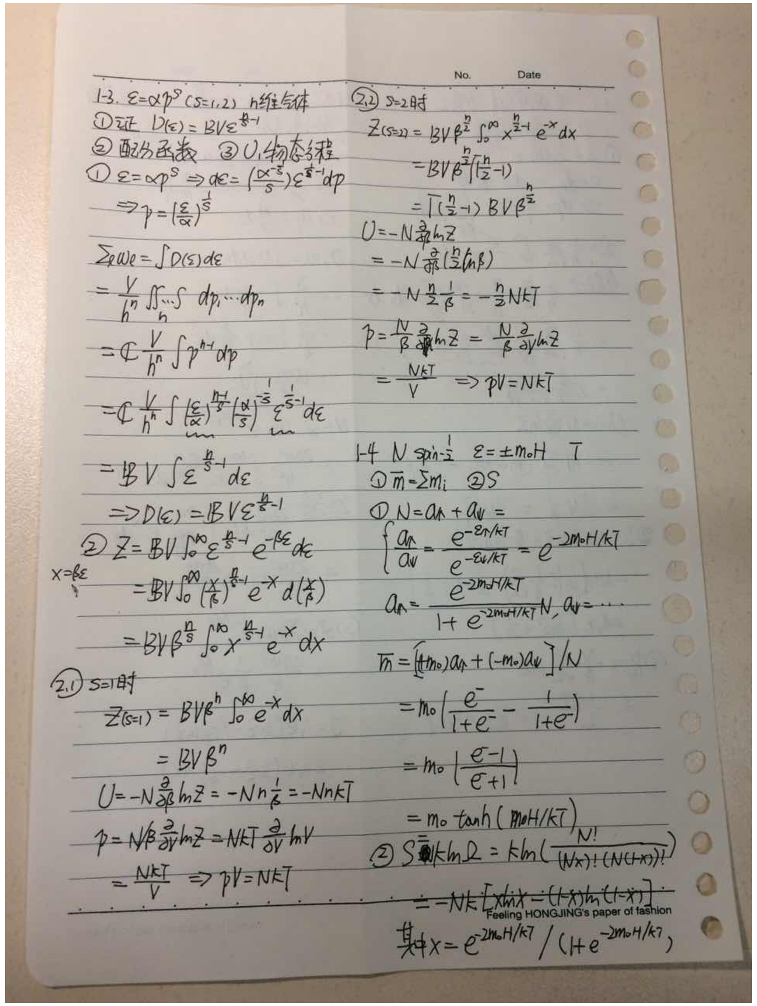
② $Z = \sum_l \omega_l e^{-\beta\varepsilon_l}$

$= \frac{2\pi mS}{h^2}\frac{1}{\beta}e^{\beta\varepsilon_0}$

$S = Nk(\ln Z - \beta\frac{\partial}{\partial\beta}\ln Z)$

$= Nk(\ln Z - \frac{\varepsilon_0}{kT} + 1)$

Feeling HONGJING's paper of fashion



No. Date

1-3. $\varepsilon = \alpha p^s\ (s=1,2)$ n维气体

① 证 $D(\varepsilon) = BV\varepsilon^{\frac{n}{s}-1}$

② 配分函数 ③ U，物态方程

① $\varepsilon = \alpha p^s \Rightarrow d\varepsilon = (\frac{\alpha^{-\frac{1}{s}}}{s})\varepsilon^{\frac{1}{s}-1}dp$

$\Rightarrow p = (\frac{\varepsilon}{\alpha})^{\frac{1}{s}}$

$\sum_l \omega_l = \int D(\varepsilon)d\varepsilon$

$= \frac{V}{h^n}\underbrace{\int\int\cdots\int}_{n} dp_1\cdots dp_n$

$= C\frac{V}{h^n}\int p^{n-1}dp$

$= C\frac{V}{h^n}\int(\frac{\varepsilon}{\alpha})^{\frac{n-1}{s}}(\frac{\alpha}{s})^{-\frac{1}{s}}\varepsilon^{\frac{1}{s}-1}d\varepsilon$

$= BV\int\varepsilon^{\frac{n}{s}-1}d\varepsilon$

$\Rightarrow D(\varepsilon) = BV\varepsilon^{\frac{n}{s}-1}$

② $Z = BV\int_0^\infty \varepsilon^{\frac{n}{s}-1}e^{-\beta\varepsilon}d\varepsilon$

$x = \beta\varepsilon$

$= BV\int_0^\infty(\frac{x}{\beta})^{\frac{n}{s}-1}e^{-x}d(\frac{x}{\beta})$

$= BV\beta^{\frac{n}{s}}\int_0^\infty x^{\frac{n}{s}-1}e^{-x}dx$

(2.1) s=1时

$Z_{(s=1)} = BV\beta^n\int_0^\infty e^{-x}dx$

$= BV\beta^n$

$U = -N\frac{\partial}{\partial\beta}\ln Z = -Nn\frac{1}{\beta} = -NnkT$

$p = N/\beta\frac{\partial}{\partial V}\ln Z = NkT\frac{\partial}{\partial V}\ln V$

$= \frac{NkT}{V} \Rightarrow pV = NkT$

(2.2) s=2时

$Z_{(s=2)} = BV\beta^{\frac{n}{2}}\int_0^\infty x^{\frac{n}{2}-1}e^{-x}dx$

$= BV\beta^{\frac{n}{2}}\Gamma(\frac{n}{2}-1)$

$= \Gamma(\frac{n}{2}-1)BV\beta^{\frac{n}{2}}$

$U = -N\frac{\partial}{\partial\beta}\ln Z$

$= -N\frac{\partial}{\partial\beta}(\frac{n}{2}\ln\beta)$

$= -N\frac{n}{2}\frac{1}{\beta} = -\frac{n}{2}NkT$

$p = \frac{N}{\beta}\frac{\partial}{\partial V}\ln Z = \frac{N}{\beta}\frac{\partial}{\partial V}\ln Z$

$= \frac{NkT}{V} \Rightarrow pV = NkT$

1-4 N spin-$\frac{1}{2}$ $\varepsilon = \pm m_0 H$ T

① $\bar{m} = \sum m_i$ ② S

① $N = a_\uparrow + a_\downarrow =$

$\frac{a_\uparrow}{a_\downarrow} = \frac{e^{-\varepsilon_\uparrow/kT}}{e^{-\varepsilon_\downarrow/kT}} = e^{-2m_0H/kT}$

$a_\uparrow = \frac{e^{-2m_0H/kT}}{1+e^{-2m_0H/kT}}N,\ a_\downarrow = \cdots$

$\bar{m} = [(+m_0)a_\uparrow + (-m_0)a_\downarrow]/N$

$= m_0(\frac{e^-}{1+e^-} - \frac{1}{1+e^-})$

$= m_0(\frac{e^- - 1}{e^- + 1})$

$= m_0\tanh(m_0H/kT)$

② $S = k\ln\Omega = k\ln(\frac{N!}{(Nx)!(N(1-x))!})$

$= -Nk[x\ln x - (1-x)\ln(1-x)]$

其中 $x = e^{-2m_0H/kT}/(1+e^{-2m_0H/kT})$

Feeling HONGJING's paper of fashion

例题

II. FERMI AND BOSE DISTRIBUTION

1. 某系统电子能态密度为

$$g(\epsilon) = \begin{cases} 0 & \epsilon < 0 \\ g_0 & \epsilon > 0 \end{cases}$$

   电子总数为N。求(a)T=0K时的化学势，总能，(b)在非简并条件下系统的化学势，总能。（04秋）

2. 求二维电子气体在0K时的费米能，内能。沿Z方向加一个B的磁场，系统会出现Pauli顺磁性，求系统磁矩。（07秋）

3. 关于简并费米气体的性质。（??）

4. 由爱因斯坦模型，求解二维系统晶格振动的自由能，熵和等容比热。（??）

5. 关于德拜模型的求解。（04春）

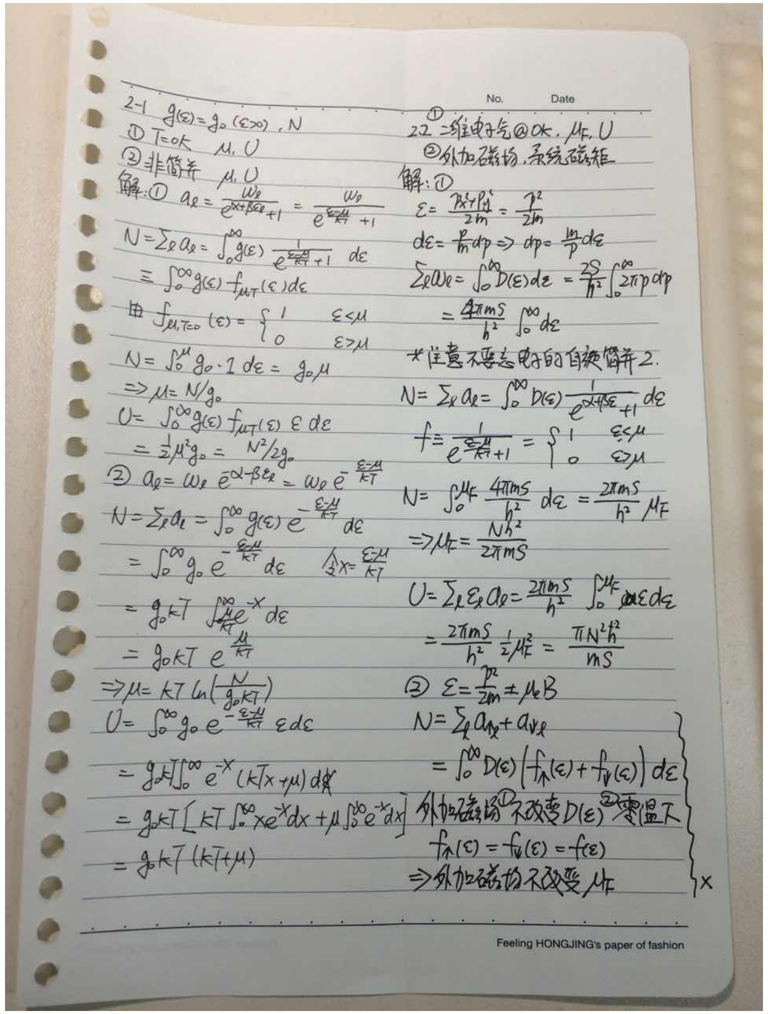
6. 关于光子气体的性质，态密度，配分函数。（04春）

例题

III. OTHERS

1. 分别划出经典粒子、光子、有质量玻色子、费米子组成的近独立粒子系统的比热随温度变化的趋势，并述原因。（04秋）
2. 描述近独立粒子体系平衡态的分布有那几种。简述各分布所对应粒子的性质。简述推导它们的主要步骤。（03秋，05春）
3. 费米气体在准静态绝热过程中压强和温度满足关系$PV^{\gamma} = Constant$。求γ的值。（07春）

解答



No. Date

2-1 $g(\varepsilon)=g_0\ (\varepsilon>0)$, N

① $T=0K$ μ, U

② 非简并 μ, U

解：① $a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}=\frac{\omega_l}{e^{\frac{\varepsilon-\mu}{kT}}+1}$

$N=\sum_l a_l=\int_0^\infty g(\varepsilon)\frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1}d\varepsilon$

$\equiv\int_0^\infty g(\varepsilon)f_{\mu,T}(\varepsilon)d\varepsilon$

由 $f_{\mu,T=0}(\varepsilon)=\begin{cases}1 & \varepsilon<\mu\\ 0 & \varepsilon>\mu\end{cases}$

$N=\int_0^\mu g_0\cdot 1\,d\varepsilon=g_0\mu$

$\Rightarrow\mu=N/g_0$

$U=\int_0^\infty g(\varepsilon)f_{\mu T}(\varepsilon)\,\varepsilon\,d\varepsilon$

$=\frac{1}{2}\mu^2 g_0=N^2/2g_0$

② $a_l=\omega_l e^{-\alpha-\beta\varepsilon_l}=\omega_l e^{-\frac{\varepsilon-\mu}{kT}}$

$N=\sum_l a_l=\int_0^\infty g(\varepsilon)e^{-\frac{\varepsilon-\mu}{kT}}d\varepsilon$

$=\int_0^\infty g_0 e^{-\frac{\varepsilon-\mu}{kT}}d\varepsilon$ 令 $x=\frac{\varepsilon-\mu}{kT}$

$=g_0kT\int_{-\frac{\mu}{kT}}^\infty e^{-x}d\varepsilon$

$=g_0kT\,e^{\frac{\mu}{kT}}$

$\Rightarrow\mu=kT\ln\left(\frac{N}{g_0kT}\right)$

$U=\int_0^\infty g_0 e^{-\frac{\varepsilon-\mu}{kT}}\varepsilon\,d\varepsilon$

$=g_0kT\int_0^\infty e^{-x}(kTx+\mu)dx$

$=g_0kT\left[kT\int_0^\infty xe^{-x}dx+\mu\int_0^\infty e^{-x}dx\right]$

$=g_0kT(kT+\mu)$

2.2 ① 二维电子气@0K, μ_F, U

② 外加磁场，系统磁矩

解：①

$\varepsilon=\frac{p_x^2+p_y^2}{2m}=\frac{p^2}{2m}$

$d\varepsilon=\frac{p}{m}dp\Rightarrow dp=\frac{m}{p}d\varepsilon$

$\sum_l\omega_l=\int_0^\infty D(\varepsilon)d\varepsilon=\frac{2S}{h^2}\int_0^\infty 2\pi p\,dp$

$=\frac{4\pi mS}{h^2}\int_0^\infty d\varepsilon$

*注意不要忘电子的自旋简并2.

$N=\sum_l a_l=\int_0^\infty D(\varepsilon)\frac{1}{e^{\alpha+\beta\varepsilon}+1}d\varepsilon$

$f=\frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1}=\begin{cases}1 & \varepsilon<\mu\\ 0 & \varepsilon>\mu\end{cases}$

$N=\int_0^{\mu_F}\frac{4\pi mS}{h^2}d\varepsilon=\frac{2\pi mS}{h^2}\mu_F$

$\Rightarrow\mu_F=\frac{Nh^2}{2\pi mS}$

$U=\sum_l\varepsilon_l a_l=\frac{2\pi mS}{h^2}\int_0^{\mu_F}\varepsilon\,d\varepsilon$

$=\frac{2\pi mS}{h^2}\frac{1}{2}\mu_F^2=\frac{\pi N^2h^2}{mS}$

② $\varepsilon=\frac{p^2}{2m}\pm\mu_e B$

$N=\sum_l a_{\uparrow l}+a_{\downarrow l}$

$=\int_0^\infty D(\varepsilon)\left(f_\uparrow(\varepsilon)+f_\downarrow(\varepsilon)\right)d\varepsilon$

外加磁场①不改变 $D(\varepsilon)$ ②零温下

$f_\uparrow(\varepsilon)=f_\downarrow(\varepsilon)=f(\varepsilon)$

$\Rightarrow$ 外加磁场不改变 μ_F } ×

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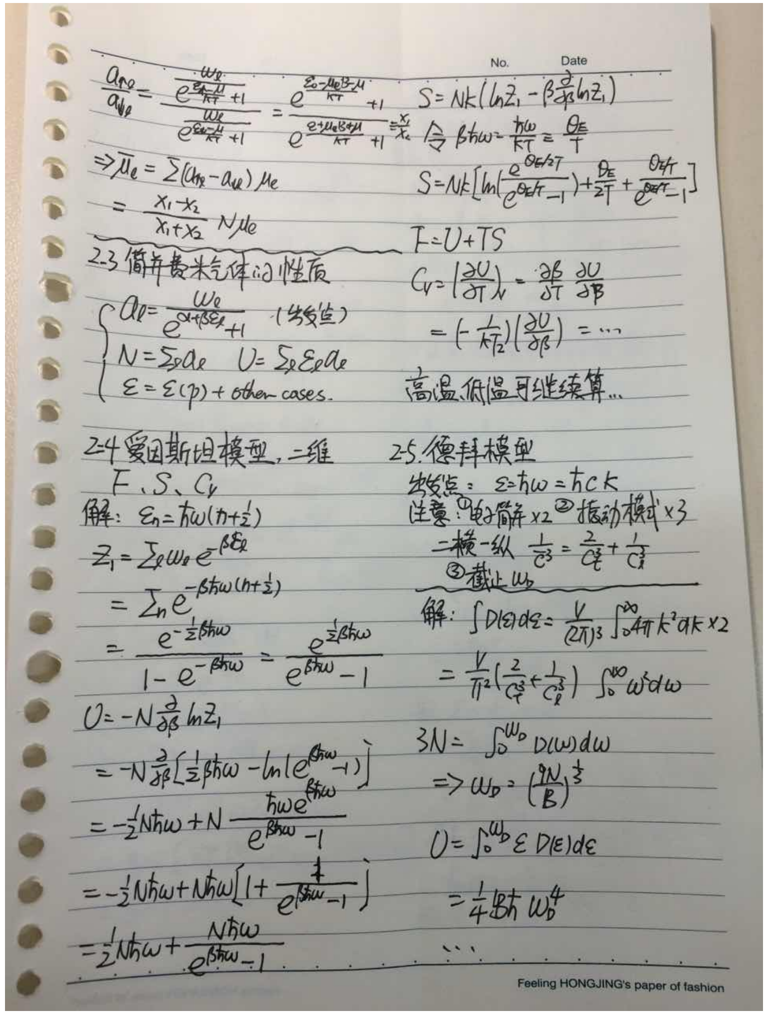

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$\frac{a_{\uparrow l}}{a_{\downarrow l}}=\frac{\frac{\omega_l}{e^{\frac{\varepsilon_\uparrow-\mu}{kT}}+1}}{\frac{\omega_l}{e^{\frac{\varepsilon_\downarrow-\mu}{kT}}+1}}=\frac{e^{\frac{\varepsilon_0-\mu_eB-\mu}{kT}}+1}{e^{\frac{\varepsilon+\mu_eB+\mu}{kT}}+1}=\frac{x_1}{x_2}$

$\Rightarrow\overline{\mu_e}=\sum(a_{\uparrow l}-a_{\downarrow l})\mu_e$

$=\frac{x_1-x_2}{x_1+x_2}N\mu_e$

2-3 简并费米气体的性质

$\begin{cases}a_l=\frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1} & \text{(出发点)}\\ N=\sum_l a_l \quad U=\sum_l\varepsilon_l a_l\\ \varepsilon=\varepsilon(p)+\text{other cases.}\end{cases}$

2-4 爱因斯坦模型，二维

F、S、C_V

解：$\varepsilon_n=\hbar\omega(n+\frac{1}{2})$

$Z_1=\sum_l\omega_l e^{-\beta\varepsilon_l}$

$=\sum_n e^{-\beta\hbar\omega(n+\frac{1}{2})}$

$=\frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}=\frac{e^{\frac{1}{2}\beta\hbar\omega}}{e^{\beta\hbar\omega}-1}$

$U=-N\frac{\partial}{\partial\beta}\ln Z_1$

$=-N\frac{\partial}{\partial\beta}\left[\frac{1}{2}\beta\hbar\omega-\ln(e^{\beta\hbar\omega}-1)\right]$

$=-\frac{1}{2}N\hbar\omega+N\frac{\hbar\omega e^{\beta\hbar\omega}}{e^{\beta\hbar\omega}-1}$

$=-\frac{1}{2}N\hbar\omega+N\hbar\omega\left[1+\frac{1}{e^{\beta\hbar\omega}-1}\right]$

$=\frac{1}{2}N\hbar\omega+\frac{N\hbar\omega}{e^{\beta\hbar\omega}-1}$

$S=Nk\left(\ln Z_1-\beta\frac{\partial}{\partial\beta}\ln Z_1\right)$

令 $\beta\hbar\omega=\frac{\hbar\omega}{kT}=\frac{\theta_E}{T}$

$S=Nk\left[\ln\left(\frac{e^{\theta_E/2T}}{e^{\theta_E/T}-1}\right)+\frac{\theta_E}{2T}+\frac{\theta_E/T}{e^{\theta_E/T}-1}\right]$

$F=U+TS$

$C_V=\left(\frac{\partial U}{\partial T}\right)_V=\frac{\partial\beta}{\partial T}\frac{\partial U}{\partial\beta}$

$=\left(-\frac{1}{kT^2}\right)\left(\frac{\partial U}{\partial\beta}\right)=\cdots$

高温、低温可继续算…

2-5. 德拜模型

出发点：$\varepsilon=\hbar\omega=\hbar ck$

注意：①电子简并 ×2 ②振动模式 ×3

二横一纵 $\frac{1}{c^3}=\frac{2}{c_t^3}+\frac{1}{c_l^3}$

③截止 ω_D

解：$\int D(\varepsilon)d\varepsilon=\frac{V}{(2\pi)^3}\int_0^\infty 4\pi k^2dk\times 2$

$=\frac{V}{\pi^2}\left(\frac{2}{c_t^3}+\frac{1}{c_l^3}\right)\int_0^\infty\omega^2d\omega$

$3N=\int_0^{\omega_D}D(\omega)d\omega$

$\Rightarrow\omega_D=\left(\frac{9N}{B}\right)^{\frac{1}{3}}$

$U=\int_0^{\omega_D}\varepsilon\,D(\varepsilon)d\varepsilon$

$=\frac{1}{4}B\hbar\,\omega_D^4$

…

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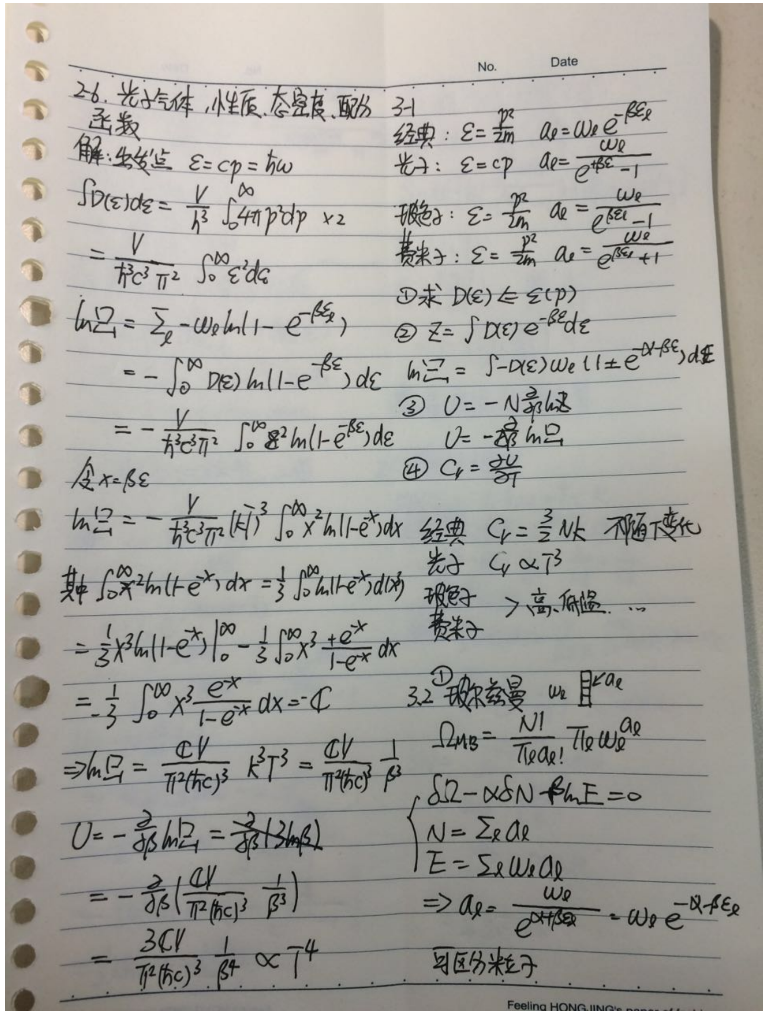
解答



2-6. 光子气体 性质、态密度、配分函数

解：光子 $\varepsilon = cp = \hbar\omega$

$\int D(\varepsilon)d\varepsilon = \frac{V}{h^3}\int_0^\infty 4\pi p^2 dp \times 2$

$= \frac{V}{\hbar^3 c^3 \pi^2}\int_0^\infty \varepsilon^2 d\varepsilon$

$\ln\Xi = \sum_l -\omega_l \ln(1-e^{-\beta\varepsilon_l})$

$= -\int_0^\infty D(\varepsilon)\ln(1-e^{-\beta\varepsilon})d\varepsilon$

$= -\frac{V}{\hbar^3 c^3\pi^2}\int_0^\infty \varepsilon^2 \ln(1-e^{-\beta\varepsilon})d\varepsilon$

令 $x=\beta\varepsilon$

$\ln\Xi = -\frac{V}{\hbar^3 c^3 \pi^2}(kT)^3\int_0^\infty x^2\ln(1-e^{-x})dx$

其中 $\int_0^\infty x^2\ln(1-e^{-x})dx = \frac{1}{3}\int_0^\infty \ln(1-e^{-x})d(x^3)$

$= \frac{1}{3}x^3\ln(1-e^{-x})\Big|_0^\infty - \frac{1}{3}\int_0^\infty x^3\frac{+e^{-x}}{1-e^{-x}}dx$

$= -\frac{1}{3}\int_0^\infty x^3\frac{e^{-x}}{1-e^{-x}}dx = -C$

$\Rightarrow \ln\Xi = \frac{CV}{\pi^2(\hbar c)^3}k^3T^3 = \frac{CV}{\pi^2(\hbar c)^3}\frac{1}{\beta^3}$

$U = -\frac{\partial}{\partial\beta}\ln\Xi = \frac{\partial}{\partial\beta}(3\ln\beta)$

$= -\frac{\partial}{\partial\beta}\left(\frac{CV}{\pi^2(\hbar c)^3}\frac{1}{\beta^3}\right)$

$= \frac{3CV}{\pi^2(\hbar c)^3}\frac{1}{\beta^4} \propto T^4$

3-1

经典：$\varepsilon = \frac{p^2}{2m}$　$a_l = \omega_l e^{-\beta\varepsilon_l}$

光子：$\varepsilon = cp$　$a_l = \frac{\omega_l}{e^{\beta\varepsilon}-1}$

玻色子：$\varepsilon = \frac{p^2}{2m}$　$a_l = \frac{\omega_l}{e^{\beta\varepsilon_l}-1}$

费米子：$\varepsilon = \frac{p^2}{2m}$　$a_l = \frac{\omega_l}{e^{\beta\varepsilon_l}+1}$

① 求 $D(\varepsilon) \Leftarrow \varepsilon(p)$

② $Z = \int D(\varepsilon)e^{-\beta\varepsilon}d\varepsilon$

$\ln\Xi = \int -D(\varepsilon)\omega_l(1\pm e^{-\alpha-\beta\varepsilon})d\varepsilon$

③ $U = -N\frac{\partial}{\partial\beta}\ln Z$

$U = -\frac{\partial}{\partial\beta}\ln\Xi$

④ $C_V = \frac{\partial U}{\partial T}$

经典 $C_V = \frac{3}{2}Nk$ 不随T变化

光子 $C_V \propto T^3$

玻色子、费米子 > 高、低温……

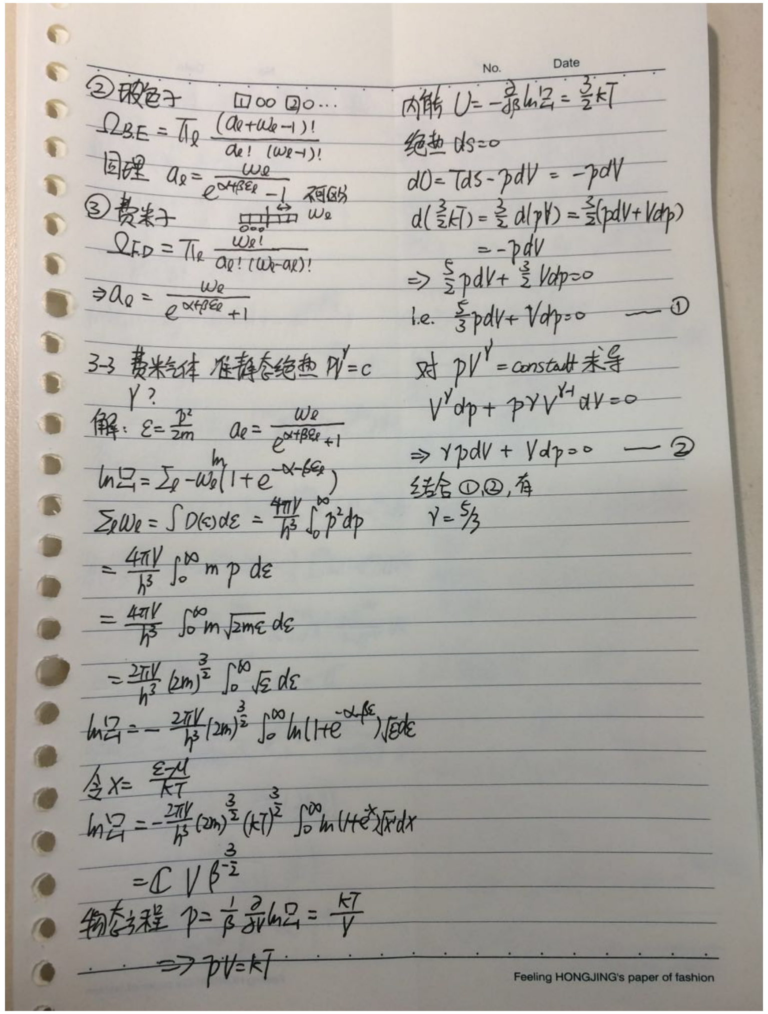
3.2 ① 玻尔兹曼 ω_l 个 a_l

$\Omega_{MB} = \frac{N!}{\prod_l a_l!}\prod_l \omega_l^{a_l}$

$\begin{cases}\delta\Omega - \alpha\delta N - \beta\delta E = 0 \\ N = \sum_l a_l \\ E = \sum_l \omega_l a_l\end{cases}$

$\Rightarrow a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}} = \omega_l e^{-\alpha-\beta\varepsilon_l}$

可区分粒子



② 玻色子　□oo □o…

$\Omega_{B.E} = \prod_l \frac{(a_l+\omega_l-1)!}{a_l!(\omega_l-1)!}$

同理 $a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1}$

③ 费米子

$\Omega_{F.D} = \prod_l \frac{\omega_l!}{a_l!(\omega_l-a_l)!}$

$\Rightarrow a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$

3-3 费米气体 准静态绝热 $pV^\gamma = c$　γ?

解：$\varepsilon = \frac{p^2}{2m}$　$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1}$

$\ln\Xi = \sum_l \omega_l\ln(1+e^{-\alpha-\beta\varepsilon_l})$

$\sum_l \omega_l = \int D(\varepsilon)d\varepsilon = \frac{4\pi V}{h^3}\int_0^\infty p^2 dp$

$= \frac{4\pi V}{h^3}\int_0^\infty m p\, d\varepsilon$

$= \frac{4\pi V}{h^3}\int_0^\infty m\sqrt{2m\varepsilon}\,d\varepsilon$

$= \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \sqrt{\varepsilon}\,d\varepsilon$

$\ln\Xi = -\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\int_0^\infty \ln(1+e^{-\alpha-\beta\varepsilon})\sqrt{\varepsilon}\,d\varepsilon$

令 $x = \frac{\varepsilon-\mu}{kT}$

$\ln\Xi = -\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}(kT)^{\frac{3}{2}}\int_0^\infty \ln(1+e^{-x})\sqrt{x}\,dx$

$= CV\beta^{-\frac{3}{2}}$

物态方程 $p = \frac{1}{\beta}\frac{\partial}{\partial V}\ln\Xi = \frac{kT}{V}$

$\Rightarrow pV = kT$

内能 $U = -\frac{\partial}{\partial\beta}\ln\Xi = \frac{3}{2}kT$

绝热 $dS = 0$

$dU = TdS - pdV = -pdV$

$d(\frac{3}{2}kT) = \frac{3}{2}d(pV) = \frac{3}{2}(pdV + Vdp)$

$= -pdV$

$\Rightarrow \frac{5}{2}pdV + \frac{3}{2}Vdp = 0$

i.e. $\frac{5}{3}pdV + Vdp = 0$ —— ①

对 $pV^\gamma = \text{constant}$ 求导

$V^\gamma dp + p\gamma V^{\gamma-1}dV = 0$

$\Rightarrow \gamma pdV + Vdp = 0$ —— ②

结合①②，有

$\gamma = 5/3$