$$\left.\begin{aligned}&\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}\\&\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\&\nabla\cdot\vec{B}=0\\&\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}\end{aligned}\right\}$$

1）关于势的基本方程

$$\vec{E}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}$$

$$\vec{B}=\nabla\times\vec{A}$$

$$\nabla^2\varphi+\frac{\partial}{\partial t}\nabla\cdot\vec{A}=-\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\left(\nabla\cdot\vec{A}+\frac{\partial\varphi}{\partial t}\right)=-\mu_0\vec{J}$$

在洛伦兹规范下： $\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$

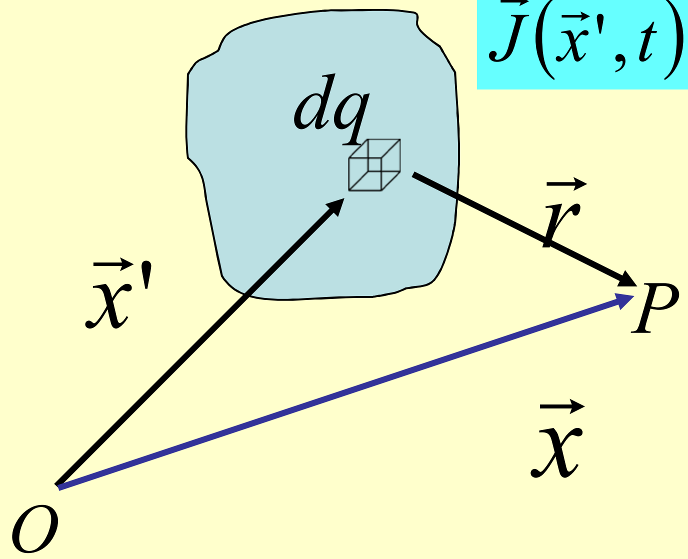
$$\left.\begin{aligned} &\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \\ &\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \end{aligned}\right\}$$ ——达朗贝尔方程

矢势和标势具有相同的形式

- 给出了空间某点 $\vec{x}$ 在 时刻的势；

$\rho(\vec{x}',t),$

$\vec{J}(\vec{x}',t)$



$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}$$

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

2）在洛伦兹规范下，对于给定的电荷电流分布，方程的解——**推迟势**

$$\rho(\vec{x}',t) \qquad \vec{J}(\vec{x}',t)$$

$$\Rightarrow \begin{cases} \varphi(\vec{x},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{1}{r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V' \\ \vec{A}(\vec{x},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V' \end{cases}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla\times\vec{A}$$

$$\Rightarrow \left(\vec{E},\vec{B}\right)$$

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

验证：推迟势满足Lorenz（规范辅助）条件：

$$\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0$$

§3 电偶极辐射

电磁辐射

- 静止电荷体系：静电场 ～$1/R^2$，无能流；
- 稳恒电流体系：电场～$1/R^2$，磁场～ $1/R^2$，能流密度～ $1/R^4$，远场总功率 ～ 0；
- 加速运动电荷体系：随时间变化的电流才会产生电磁波——辐射场(远场总功率不为零)

- 交变运动的电荷系统辐射出电磁波；
- 本节主要研究**宏观的电荷系统**在其线度远小于辐射波长情况的辐射问题

本节主要内容

1. 计算辐射场的一般公式
2. 矢势的展开
3. 偶极辐射
4. 偶极辐射的能流、角分布、功率
5. 短天线的辐射

1. 计算辐射场的一般公式

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

3）给定交变电流分布下的矢势解

假设： $\vec{J}(\vec{x}',t)=\vec{J}(\vec{x}')\mathrm{e}^{-i\omega t}$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}(\vec{x}')\mathrm{e}^{-\mathrm{i}\omega\left(t-\frac{r}{c}\right)}\mathrm{d}V'$$

$$=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}(\vec{x}')\mathrm{e}^{\mathrm{i}\left(\frac{\omega}{c}r-\omega t\right)}\mathrm{d}V'$$

$$=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\mathrm{d}V'$$

$$k=\omega/c=\omega\sqrt{\mu_0\varepsilon_0}$$

—真空中的波数

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\left[\int\frac{\vec{J}(\vec{x}')}{r}\mathrm{e}^{\mathrm{i}kr}\mathrm{d}V'\right]\mathrm{e}^{-\mathrm{i}\omega t}$$

$$=\vec{A}(\vec{x})\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\Rightarrow\quad \vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}\mathrm{e}^{\mathrm{i}kr}\mathrm{d}V'$$

① 因子 e^{ikr} 为推迟作用因子，表示电磁波传至场点需要一定的时间，从而在位相上滞后 $kr=2\pi r/\lambda$。

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}\mathrm{e}^{\mathrm{i}kr}\mathrm{d}V'$$

$$\Rightarrow \quad \vec{B}=\nabla\times\vec{A}$$

② 电荷分布区域以外（无电流分布），可以由磁场求出电场

$$\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$$

$$\nabla\times\vec{B}(\vec{x},t)=\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{x},t)$$

$$\vec{E}(\vec{x},t)=\vec{E}(\vec{x})\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\Rightarrow \quad \nabla\times\vec{B}(\vec{x})=-\frac{\mathrm{i}\omega}{c^2}\vec{E}(\vec{x})$$

$$\vec{E}(\vec{x})=\frac{\mathrm{i}c}{k}\nabla\times\vec{B}(\vec{x})$$

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

③ 交变电流下的标势解

电荷密度与电流密度满足电荷守恒定律；

$$\vec{J}(\vec{x}',t)=\vec{J}(\vec{x}')\mathrm{e}^{-i\omega t}$$

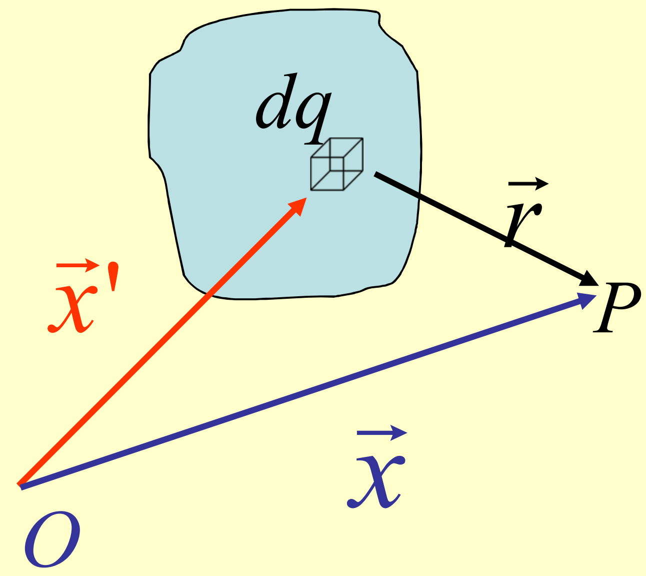
$$\rho(\vec{x}',t)=\rho(\vec{x}')\mathrm{e}^{-i\omega t}$$

$$\nabla'\cdot\vec{J}(\vec{x}',t)+\frac{\partial\rho(\vec{x}',t)}{\partial t}=0$$

$$\mathrm{i}\omega\rho(\vec{x}')=\nabla'\cdot\vec{J}(\vec{x}')$$

$$\rho(\vec{x}')=-\mathrm{i}\omega^{-1}\nabla'\cdot\vec{J}(\vec{x}')$$

2、矢势的展开式



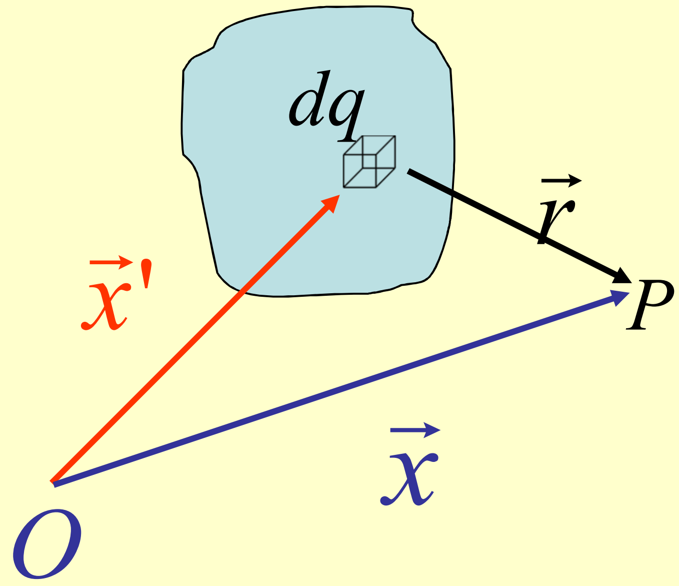
假设源点： $\vec{J}(\vec{x}',t)=\vec{J}(\vec{x}')e^{-i\omega t}$

矢势： $\vec{A}(\vec{x},t)=\vec{A}(\vec{x})e^{-i\omega t}$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}e^{ikr}\,dV'$$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}e^{ikr}dV'$$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{r}e^{i\frac{2\pi}{\lambda}r}dV'$$



积分中牵涉到三个尺度：

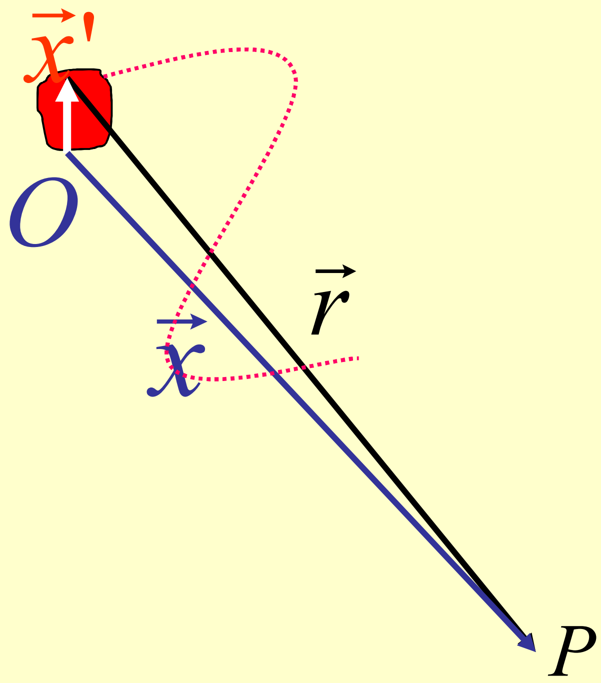
① 电流分布的区域线度 l；

② 波长 λ ；

③ 电流分布区域到场点的距离 r

1）小区域电流体系所产生的辐射问题

① **小区域电荷体系指**：电荷分布区域线度 l 远小于观测距离距离 r 和波长 λ。

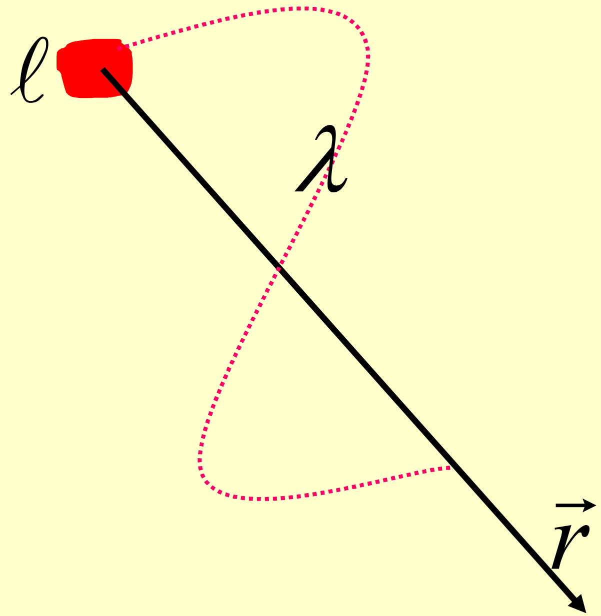


$$\ell << \lambda$$

$$\ell << r$$

$$\ell << \lambda$$

$$\ell << r$$



② 根据 λ 和 r 之间的相对关系，又可分成以下三种情况

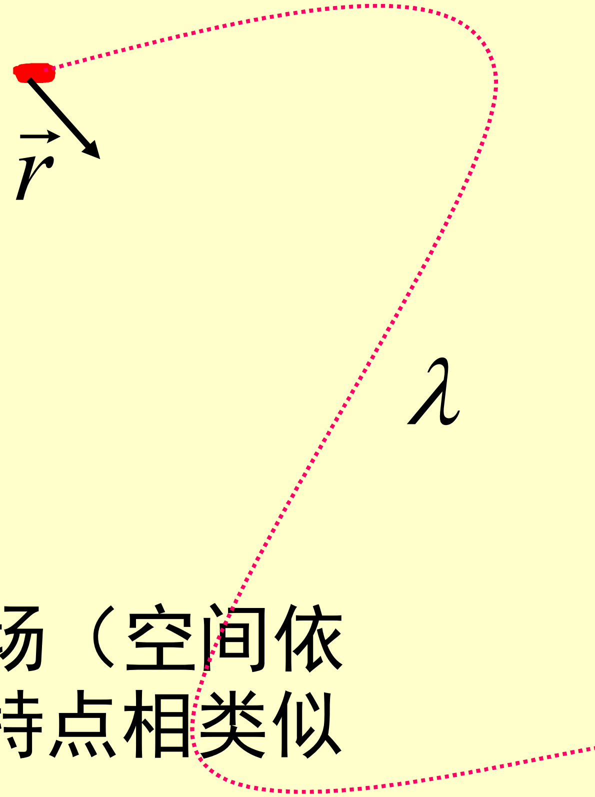
小区域电流情况，根据场所在位置

$r << \lambda$　　　$r \sim \lambda$　　　$r >> \lambda$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}\,')}{r}e^{i\frac{2\pi}{\lambda}r}dV'$$

A）近区场

$$\left(\frac{r}{\lambda}<<1\right)$$



$$\frac{2\pi}{\lambda}r=kr<<1$$

$$e^{ikr}\sim 1$$

在近场区，推迟效应可忽略，电磁场（空间依赖因子部分）与恒场（静场）的特点相类似

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}\,')}{r}dV'$$

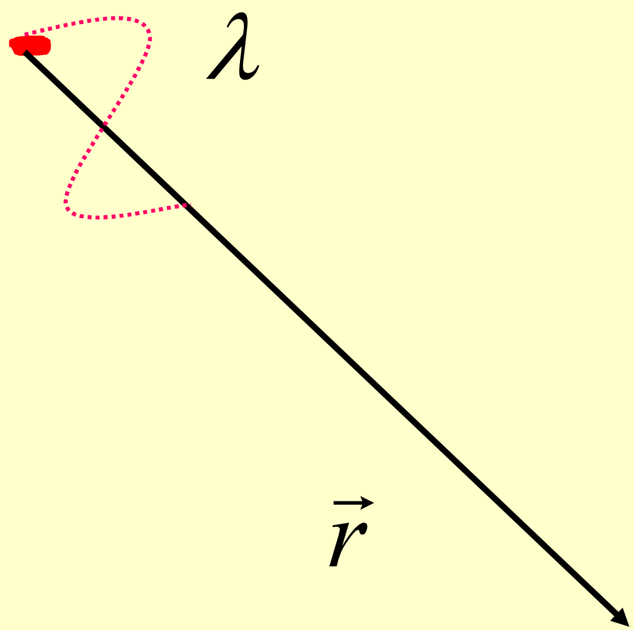
$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{r} e^{i\frac{2\pi}{\lambda}r} dV'$$

B）感应区场

$r \sim \lambda$ ，亦称为过度区

C）辐射区场

$r >> \lambda$ ，亦称之为远区



- 在远离发射系统接受电磁波，则需要讨论远场问题；
- 研究场对电荷系统的影响，则需要讨论近区和感应区的电磁场。

2）**远区电磁场　矢势的展开**

对于远场区域的辐射场，可以采用近似方法求解

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{r} e^{i\frac{2\pi}{\lambda}r} dV'$$

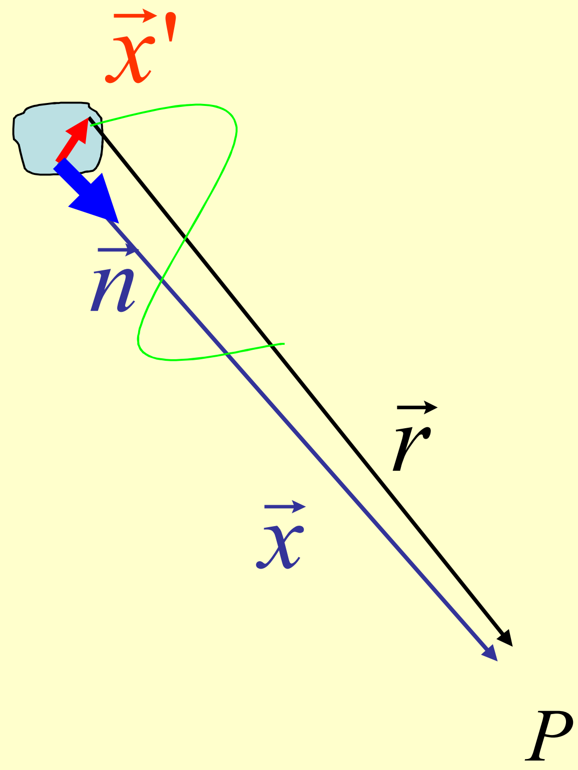
$$R = |\vec{x}|$$

将坐标的原点选在电荷分布的区域内 $(|\vec{x}'| \sim \ell)$

$$r = \left(\vec{x}^2 - 2\vec{x}\cdot\vec{x}' + \vec{x}'^2\right)^{1/2}$$

$$\approx R\left(1 - 2\frac{\vec{x}}{R}\cdot\vec{x}'\right)^{1/2}$$

$$\approx R - \vec{n}\cdot\vec{x}'$$



$$r \approx R - \vec{n}\cdot\vec{x}'$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{r} e^{i\frac{2\pi}{\lambda}r} dV'$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{R-\vec{n}\cdot\vec{x}'} e^{i\frac{2\pi}{\lambda}(R-\vec{n}\cdot\vec{x}')} dV'$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{R-\vec{n}\cdot\vec{x}'} e^{i2\pi\left(\frac{R}{\lambda}-\frac{\vec{n}\cdot\vec{x}'}{\lambda}\right)} dV'$$

$R-\vec{n}\cdot\vec{x}' \approx R$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{i\frac{2\pi}{\lambda}R}}{R}\int \vec{J}(\vec{x}') e^{-i\frac{2\pi}{\lambda}\vec{n}\cdot\vec{x}'} dV'$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}R}}{R}\int \vec{J}(\vec{x}')\mathrm{e}^{-\mathrm{i}\frac{2\pi}{\lambda}\vec{n}\cdot\vec{x}'}\mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\int \vec{J}(\vec{x}')\,\mathrm{e}^{-\mathrm{i}k\vec{n}\cdot\vec{x}'}\mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\int \vec{J}(\vec{x}')[1-\mathrm{i}k\,\vec{n}\cdot\vec{x}'+......]\mathrm{d}V'$$

$$\mathrm{e}^{-\mathrm{i}\frac{2\pi}{\lambda}\vec{n}\cdot\vec{x}'} = \mathrm{e}^{-\mathrm{i}k\vec{n}\cdot\vec{x}'} = 1-\mathrm{i}k\vec{n}\cdot\vec{x}'+......$$ 小区域电流分布

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikR}}{R} \int \vec{J}(\vec{x}') \left[1 - ik\, \vec{n} \cdot \vec{x}' + \ldots\ldots\right] dV'$$

3. 偶极辐射（dipole radiation）

展开式中的各项对应电磁多极辐射

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\frac{e^{ikR}}{R}\int\vec{J}(\vec{x}')[1-ik\,\vec{n}\cdot\vec{x}'+......]dV'$$

在考虑辐射场时

$$A^{(0)}(\vec{x})=\frac{\mu_0}{4\pi}\frac{e^{ikR}}{R}\int\vec{J}(\vec{x}')dV'$$

- 一般的情况下，总是以第一个非零项为主；
- 如果$A^{(0)}\neq 0$，则只考虑$A^{(0)}$，其它的项忽略；
- 如果$A^{(0)}=0$，则需要考虑$A^{(1)}$ 等更高阶的项。

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikR}}{R}\int \vec{J}(\vec{x}')\left[1 - ik\,\vec{n}\cdot\vec{x}' + \ldots\ldots\right]dV'$$

1）零级项：$A^{(0)}(\vec{x}) = \dfrac{\mu_0}{4\pi}\dfrac{e^{ikR}}{R}\int \vec{J}(\vec{x}')dV'$

采用分步积分法，可以证明：

$$\int \vec{J}(\vec{x}')dV' \Rightarrow -\int \vec{x}'\nabla'\cdot\vec{J}(\vec{x}')dV'$$

$$\rho(\vec{x}') = -i\omega^{-1}\nabla'\cdot\vec{J}(\vec{x}')$$

$$\nabla\cdot(\vec{f}\vec{g}) = (\nabla\cdot\vec{f})\vec{g} + (\vec{f}\cdot\nabla)\vec{g}$$ (—— I.53)

$$\oint \mathrm{d}\vec{S}\cdot(\vec{f}\,\vec{g}) = \int \nabla\cdot(\vec{f}\,\vec{g})\,\mathrm{d}V$$ (—— I.52)

$$= \int \mathrm{d}V(\nabla\cdot\vec{f})\vec{g} + \int \mathrm{d}V(\vec{f}\cdot\nabla)\vec{g}$$

$$\oint \mathrm{d}\vec{S}'\cdot(\vec{J}\vec{x}') = \int \mathrm{d}V'(\nabla'\cdot\vec{J})\vec{x}' + \int \mathrm{d}V'(\vec{J}\cdot\nabla')\vec{x}'$$

$$(\vec{J}\cdot\nabla')\vec{x}' = \left(J_x\frac{\partial}{\partial x'} + J_y\frac{\partial}{\partial y'} + J_z\frac{\partial}{\partial z'}\right)\left(x'\vec{e}_{x'} + y'\vec{e}_{y'} + z'\vec{e}_{z'}\right) = \vec{J}$$

$$= \int \mathrm{d}V'(\nabla'\cdot\vec{J})\,\vec{x}' + \int \mathrm{d}V'\,\vec{J}$$

$$\oint d\vec{S}'\cdot(\vec{J}\,\vec{x}') = \int dV'(\nabla'\cdot\vec{J})\,\vec{x}' + \int dV'\,\vec{J}$$

左边 $\oint d\vec{S}'\cdot(\vec{J}\,\vec{x}') = \oint (d\vec{S}'\cdot\vec{J})\,\vec{x}' = 0$

因为积分区域V内包含所有的 $\vec{J}$ 所以在积分区域边界面 S' 的法向分量为零。

$$\vec{J}(\vec{x}', t) = \vec{J}(\vec{x}')e^{-i\omega t}$$

$$\rho(\vec{x}', t) = \rho(\vec{x}')e^{-i\omega t}$$

$$i\omega\rho(\vec{x}') = \nabla'\cdot\vec{J}(\vec{x}')$$

$$\Rightarrow \int dV'(\nabla'\cdot\vec{J})\,\vec{x}' + \int dV'\,\vec{J} = 0$$

$$\int \vec{J}(\vec{x}')dV' = -\int \vec{x}'[\nabla'\cdot\vec{J}(\vec{x}')]dV'$$

$$= -\int \vec{x}'\,i\omega\rho(\vec{x}')dV'$$

$$= -i\omega\int \vec{x}'\rho(\vec{x}')dV'$$

$$= -i\omega\vec{p}$$

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')dV'$$

$$\int \vec{J}(\vec{x}')\mathrm{d}V' = -\mathrm{i}\omega\vec{p}$$

$\vec{p}$ 为电荷系统的电偶极矩（仅含振幅部分）

$$A^{(0)}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\int \vec{J}(\vec{x}')\mathrm{d}V' = \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}(-\mathrm{i}\omega\vec{p})$$

考虑时间振荡因子，则

$$\vec{A}^{(0)}(\vec{x},t) = \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}(-\mathrm{i}\omega\vec{p})\mathrm{e}^{-\mathrm{i}\omega t}$$

$$= \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\dot{\vec{p}}$$

其中定义：$\dot{\vec{p}} = -\mathrm{i}\omega\vec{p}\mathrm{e}^{-\mathrm{i}\omega t}$

$$A^{(0)}(\vec{x},t)=\frac{\mu_0}{4\pi}\frac{e^{ikR}}{R}\dot{\vec{p}}$$

2）磁场

$$\vec{B}^{(0)}=\nabla\times A^{(0)}=\nabla\times\left[\frac{\mu_0}{4\pi}\frac{e^{ikR}}{R}\dot{\vec{p}}\right]$$

$$=\frac{\mu_0}{4\pi}\nabla\left(\frac{e^{ikR}}{R}\right)\times\dot{\vec{p}}$$

$$\nabla\times\left(\varphi\vec{f}\right)=(\nabla\varphi)\times\vec{f}+\varphi\nabla\times\vec{f}$$

$$\nabla\left(\frac{e^{ikR}}{R}\right)=\left(-\frac{1}{R^2}\vec{e}_r+ik\frac{1}{R}\vec{e}_r\right)e^{ikR}\approx ik\frac{e^{ikR}}{R}\vec{n}$$

$$\nabla\psi=\frac{\partial\psi}{\partial R}\vec{e}_r+\frac{1}{R}\frac{\partial\psi}{\partial\theta}\vec{e}_\theta+\frac{1}{R\sin\theta}\frac{\partial\psi}{\partial\varphi}\vec{e}_\phi$$

$$\nabla\left(\frac{e^{ikR}}{R}\right)\approx ik\frac{e^{ikR}}{R}\vec{n}$$

$$\vec{B}^{(0)}=\frac{\mu_0}{4\pi}\nabla\left(\frac{e^{ikR}}{R}\right)\times\dot{\vec{p}}=\frac{i\mu_0 k}{4\pi}\frac{e^{ikR}}{R}\vec{n}\times\dot{\vec{p}}$$

$$=\frac{i\mu_0 k}{4\pi}\left(\frac{1}{-i\omega}\right)\frac{e^{ikR}}{R}\vec{n}\times\left(-i\omega\dot{\vec{p}}\right)$$

定义：$\ddot{\vec{p}}=-i\omega\dot{\vec{p}}=-\omega^2\vec{p}e^{-i\omega t}$

$$\vec{B}^{(0)}=-\frac{\mu_0 k}{4\pi\omega}\frac{e^{ikR}}{R}\vec{n}\times\ddot{\vec{p}}$$

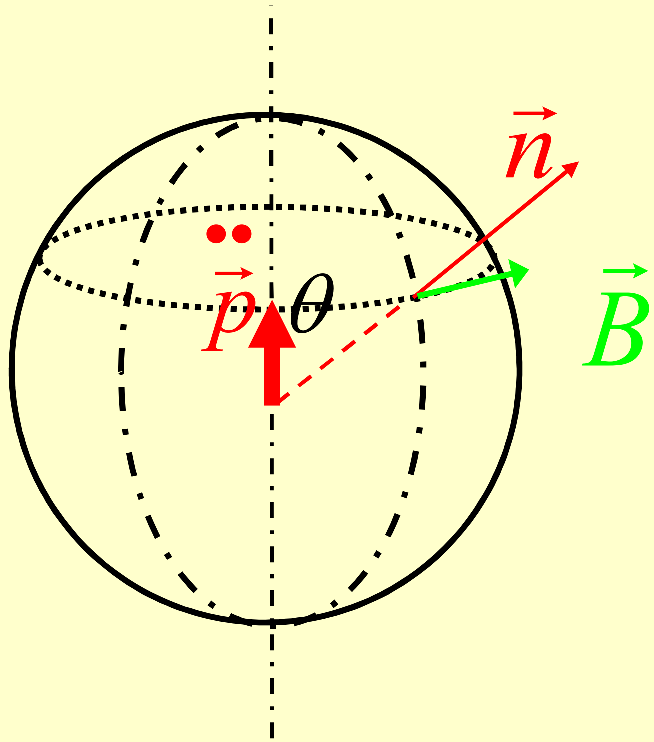
$$\vec{B}^{(0)} = -\frac{\mu_0 k}{4\pi\omega}\frac{e^{ikR}}{R}\vec{n}\times\ddot{\vec{p}}$$

$$\vec{B}^{(0)} = \frac{\mu_0}{4\pi c}\frac{e^{ikR}}{R}\ddot{\vec{p}}\times\vec{n}$$

$$= \frac{1}{4\pi\varepsilon_0 c^3}\frac{e^{ikR}}{R}\ddot{\vec{p}}\times\vec{n}$$

$$= \frac{1}{4\pi\varepsilon_0 c^3}\frac{e^{ikR}}{R}\ddot{p}\sin\theta\,\vec{e}_\phi$$

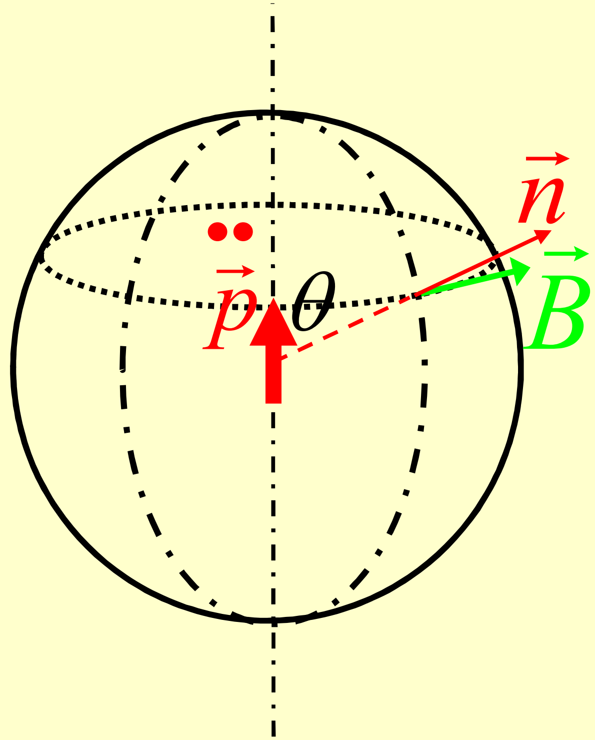
$$= B_\phi(R,\theta)\vec{e}_\phi$$



$$B_\phi(R,\theta) = \frac{1}{4\pi\varepsilon_0 c^3}\ddot{p}\frac{e^{ikR}}{R}\sin\theta \quad = C_1\frac{e^{ikR}}{R}\sin\theta$$

2）电场 $\vec{E} = \frac{ic}{k}\nabla \times \vec{B}_\phi(R,\theta)$

$$\vec{B}^{(0)} = C_1 \frac{e^{ikR}}{R}\sin\theta\,\vec{e}_\phi$$



$$\vec{E} = \frac{ic}{k}\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(C_1\frac{e^{ikR}}{R}\sin^2\theta\right)\vec{e}_R$$

$$\propto \frac{1}{R^2}$$

$$+\frac{ic}{k}\frac{-1}{R}\frac{\partial}{\partial R}\left(C_1 e^{ikR}\sin\theta\right)\vec{e}_\theta$$

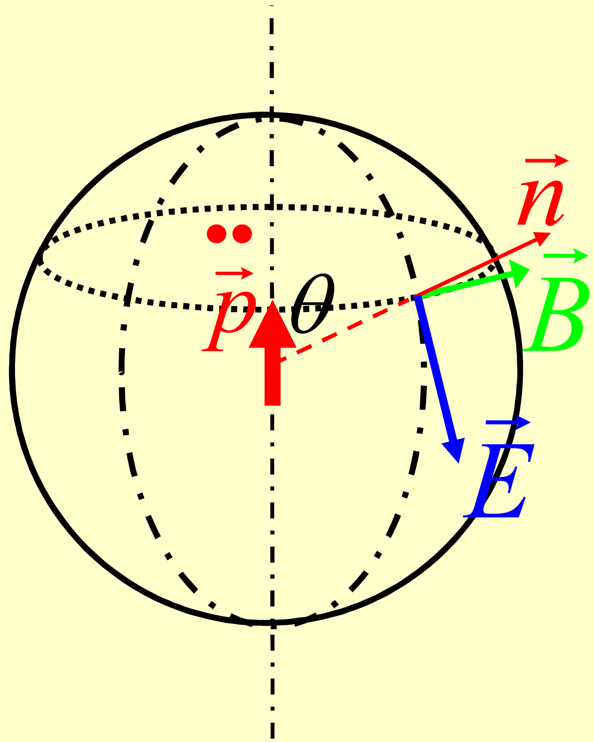
$$\nabla\times\vec{f} = \nabla\times\left[f_\phi\vec{e}_\phi\right] = \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta f_\phi\right)\vec{e}_R - \frac{1}{R}\frac{\partial}{\partial R}\left(Rf_\phi\right)\vec{e}_\theta$$

$$C_1 = \frac{1}{4\pi\varepsilon_0 c^3}\ddot{p}$$

$$\vec{E} \approx \frac{\mathrm{i}c}{k}\frac{-1}{R}\frac{\partial}{\partial R}\left(C_1 \mathrm{e}^{\mathrm{i}kR}\sin\theta\right)\vec{\mathrm{e}}_\theta$$

$$\approx \frac{\mathrm{i}c}{k}\frac{1}{4\pi\varepsilon_0 c^3}\ddot{p}\frac{-\mathrm{i}\,k}{R}\mathrm{e}^{\mathrm{i}kR}\sin\theta\,\vec{\mathrm{e}}_\theta$$

$$= \frac{1}{4\pi\varepsilon_0 c^2 R}\mathrm{e}^{\mathrm{i}kR}\ddot{p}\sin\theta\,\vec{\mathrm{e}}_\theta$$



$$\ddot{\vec{p}} = -\mathrm{i}\omega\,\dot{\vec{p}} = -\omega^2\,\vec{p}\,\mathrm{e}^{-\mathrm{i}\omega t}$$

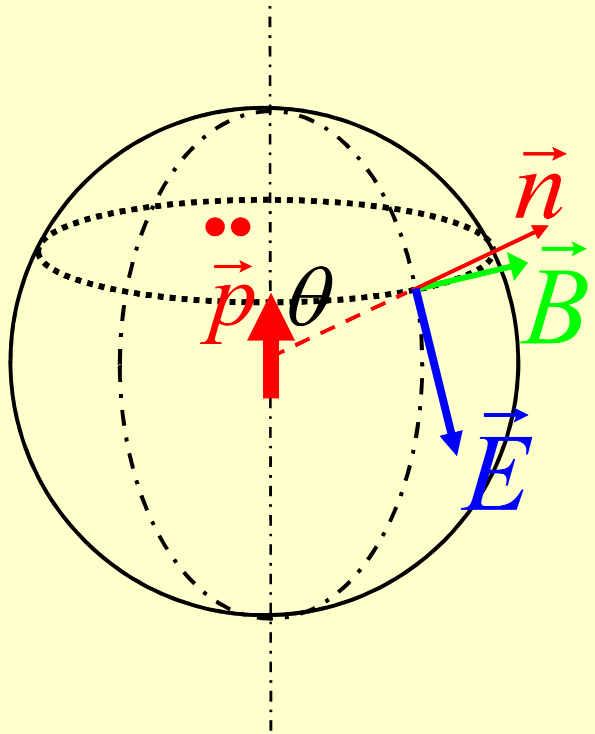
考虑时间振荡因子: $\mathrm{e}^{-i\omega t}$

$$\vec{B} = \frac{-1}{4\pi\varepsilon_0 c^3}\frac{1}{R}\left|\ddot{\vec{p}}\right|\sin\theta\ \mathrm{e}^{\mathrm{i}(kR-\omega t)}\vec{\mathrm{e}}_\varphi$$

$$\vec{E} = \frac{-1}{4\pi\varepsilon_0 c^2}\frac{1}{R}\left|\ddot{\vec{p}}\right|\sin\theta\ \mathrm{e}^{\mathrm{i}(kR-\omega t)}\vec{\mathrm{e}}_\theta$$

$$\vec{B} = \frac{1}{4\pi\varepsilon_0 c^3}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\ddot{p}\sin\theta\,\vec{\mathrm{e}}_\phi$$

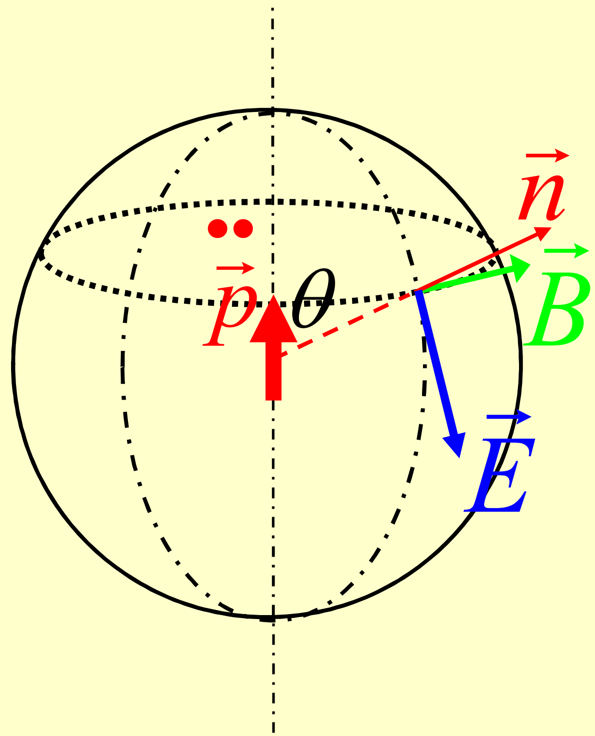
$$\vec{E} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\ddot{p}\sin\theta\,\vec{\mathrm{e}}_\theta$$



总结： $\ddot{\vec{p}}=-\omega^2\vec{p}\mathrm{e}^{-\mathrm{i}\omega t}$ $\left|\ddot{\vec{p}}\right|=\omega^2\left|\vec{p}\right|$

$$\vec{B}=\frac{-1}{4\pi\varepsilon_0 c^3}\frac{1}{R}\left|\ddot{\vec{p}}\right|\sin\theta\ \mathrm{e}^{\mathrm{i}(kR-\omega t)}\vec{\mathrm{e}}_\varphi$$

$$\vec{E}=\frac{-1}{4\pi\varepsilon_0 c^2}\frac{1}{R}\left|\ddot{\vec{p}}\right|\sin\theta\mathrm{e}^{\mathrm{i}(kR-\omega t)}\ \vec{\mathrm{e}}_\theta$$



① B线沿纬度线上振荡；E线沿经度线上振荡；

② 远场区，**电场和磁场振幅都具有 1/R 的特点**（*不同于静电场和静磁场*）；

③ 具有这种特性的场，在运动中伴随有能量的辐射，这样的场称为**辐射场**。

4. **时变电偶极矩**在远场区激发的电磁场辐射

——能流、辐射功率、角分布

1）平均能流密度：

$$\vec{B}=\frac{1}{4\pi\varepsilon_0 c^3}\frac{1}{R}\ddot{p}\sin\theta\ \mathrm{e}^{\mathrm{i}(kR-\omega t)}\vec{\mathrm{e}}_\varphi$$

$$\vec{E}=\frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{R}\ddot{p}\sin\theta\mathrm{e}^{\mathrm{i}(kR-\omega t)}\ \vec{\mathrm{e}}_\theta$$

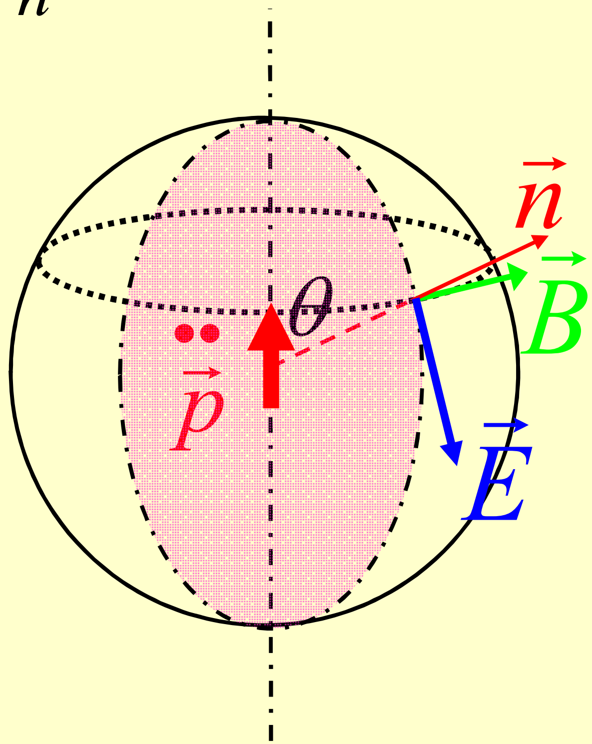
$$\vec{S}=\vec{E}\times\vec{H}=\mathrm{Re}\left(\vec{E}\right)\times\mathrm{Re}\left(\vec{H}\right)$$

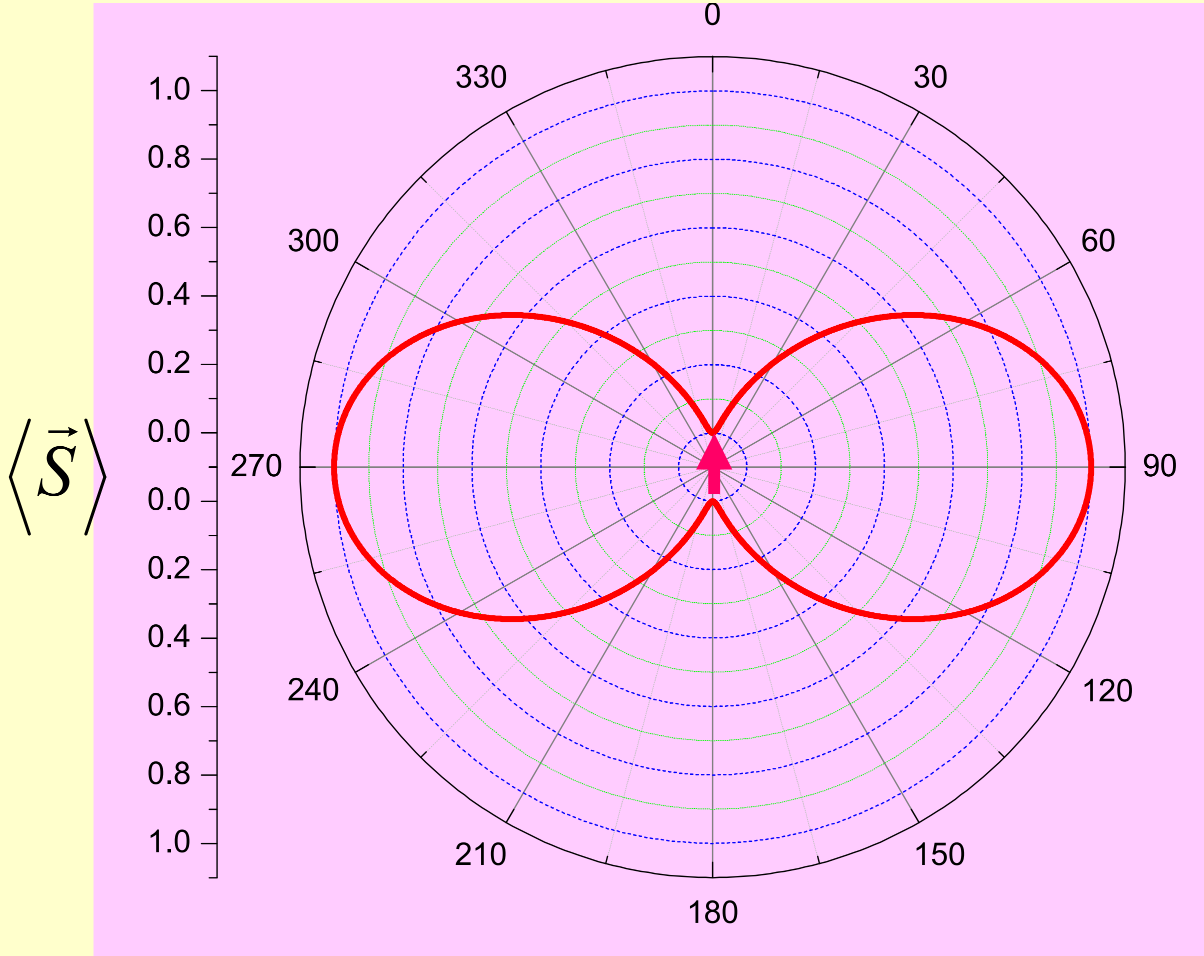
$$=\mathrm{Re}\left(\vec{E}\right)\times\frac{1}{\mu_0}\mathrm{Re}\left(\vec{B}\right)$$

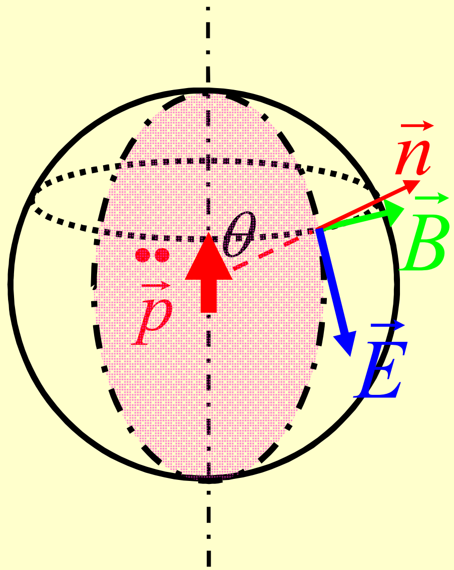
$$=\frac{1}{16\mu_0\pi^2\varepsilon_0^2c^5}\frac{1}{R^2}\left|\ddot{p}\right|^2\sin^2\theta\left[\cos\left(kR-\omega t\right)\right]^2\vec{n}$$

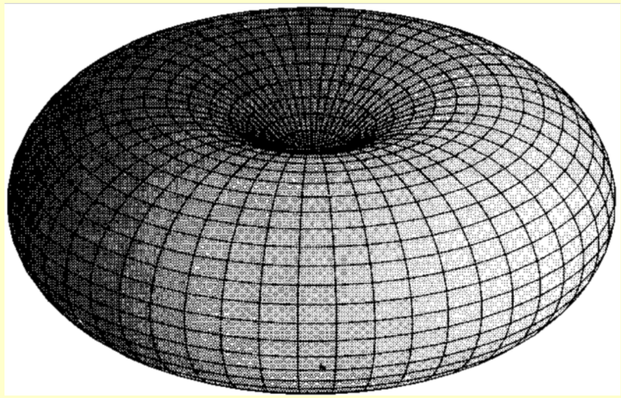
$$\left\langle\vec{S}\right\rangle=\frac{1}{32\mu_0\pi^2\varepsilon_0^2c^5}\frac{1}{R^2}\left|\ddot{p}\right|^2\sin^2\theta\ \vec{n}$$

$$=\frac{1}{32\pi^2\varepsilon_0 c^3}\frac{1}{R^2}\left|\ddot{p}\right|^2\sin^2\theta\ \vec{n}$$









偶极子辐射：

- 在电偶极矩的轴线方向上没有辐射；
- 在ϑ=90°的平面上的辐射最强。

$$\langle\vec{S}\rangle = \frac{1}{32\pi^2\varepsilon_0 c^3}\frac{1}{R^2}\left|\ddot{p}\right|^2 \sin^2\theta\ \vec{n}$$

2）总辐射功率

总辐射功率为平均能流密度对球面积分

$$P = \oint \left|\langle\vec{S}\rangle\right| R^2 \mathrm{d}\Omega$$

$$= \frac{1}{32\pi^2\varepsilon_0 c^3}\left|\ddot{p}\right|^2 \oiint \sin^2\theta\ \mathrm{d}\Omega$$

$$= \frac{1}{32\pi^2\varepsilon_0 c^3}\left|\ddot{p}\right|^2 \int_0^{\pi} \sin^3\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{3c^3}\left|\ddot{p}\right|^2$$

$$P=\frac{1}{4\pi\varepsilon_0}\frac{1}{3c^3}\left|\ddot{p}\right|^2$$

$$\ddot{\vec{p}}=-\mathrm{i}\omega\dot{\vec{p}}=-\omega^2\vec{p}\mathrm{e}^{-\mathrm{i}\omega t}$$

时变电偶极矩在远场区辐射的特点：

① 远场区球面上的总辐射功率与距离无关，因此电磁波可以传播很远；

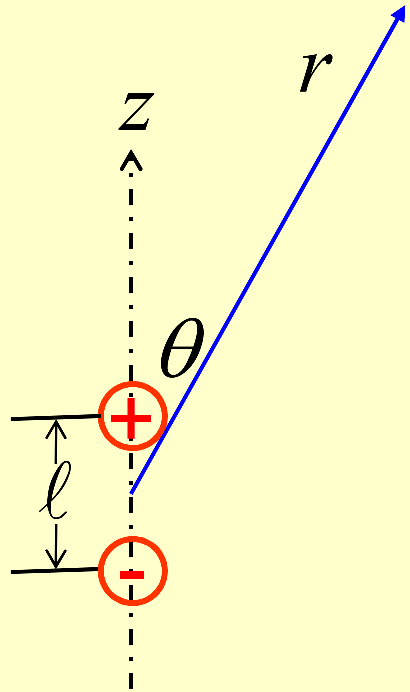
② 如果保持电偶极矩的振幅不变，频率增高时，辐射功率迅速增大。

例题：

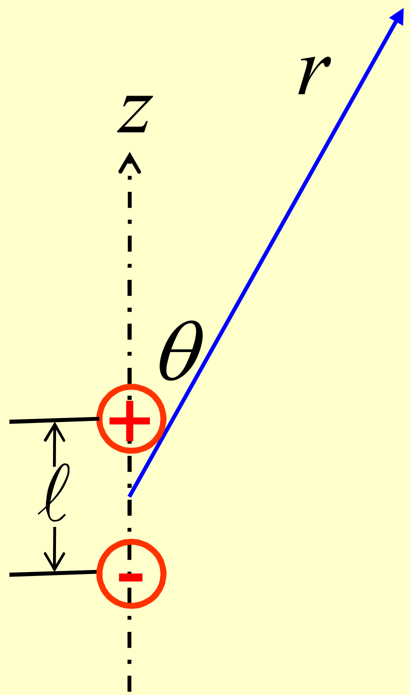
假设两个很小的金属球，之间用细导线相连，两个球上的电量分别是q(t)和-q(t)。假设：

$$q(t) = q_0 \cos(\omega t)$$

计算平均能流密度和总辐射功率。



偶极子辐射



假设两个很小的金属球，之间用细导线相连，两个球上的电量分别是q(t)和-q(t)。假设：

$$q(t) = q_0 \cos(\omega t)$$

电偶极矩：$\vec{p}(t) = q_0 \ell \cos(\omega t) \vec{e}_z$

$$= p_0 \cos(\omega t) \vec{e}_z$$

$$\vec{p}(t) = p_0 e^{-i\omega t} \vec{e}_z \quad (p_0 = q_0 \ell)$$

$$\Rightarrow \dot{\vec{p}} = -i\omega p_0 e^{-i\omega t} \vec{e}_z$$

$$\Rightarrow \ddot{\vec{p}} = -i\omega \dot{\vec{p}} = -\omega^2 p_0 e^{-i\omega t} \vec{e}_z$$

$$\vec{B} = \frac{1}{4\pi\varepsilon_0 c^3} \frac{1}{R} \ddot{p} \sin\theta \, e^{i(kR-\omega t)} \vec{e}_\varphi$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 c^2} \frac{1}{R} \ddot{p} \sin\theta e^{i(kR-\omega t)} \vec{e}_\theta$$

$$\ddot{\vec{p}} = -\omega^2 p_0 e^{-i\omega t} \vec{e}_z$$

$$\vec{B} = \frac{-1}{4\pi\varepsilon_0 c^3} \frac{1}{R} \omega^2 p_0 \sin\theta \, e^{i(kR-\omega t)} \vec{e}_\phi$$

$$= \frac{-\mu_0 p_0 \omega^2}{4\pi c} \frac{\sin\theta}{R} e^{i(kR-\omega t)} \vec{e}_\phi$$

$$\vec{E} = \frac{-1}{4\pi\varepsilon_0 c^2} \frac{1}{R} \omega^2 p_0 \sin\theta e^{i(kR-\omega t)} \vec{e}_\theta$$

$$= \frac{-\mu_0 \omega^2 p_0}{4\pi} \frac{\sin\theta}{R} e^{i(kR-\omega t)} \vec{e}_\theta$$

$$\langle \vec{S} \rangle = \frac{\mu_0 \omega^4 p_0^2}{32\pi^2 c} \frac{\sin^2\theta}{R^2} \vec{n}$$

$$P = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

$$\vec{B} = \frac{1}{4\pi\varepsilon_0 c^3} \frac{1}{R} \ddot{p} \sin\theta \, e^{i(kR-\omega t)} \vec{e}_\phi$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 c^2} \frac{1}{R} \ddot{p} \sin\theta e^{i(kR-\omega t)} \vec{e}_\theta$$

$$\langle \vec{S} \rangle = \frac{1}{32\pi^2 \varepsilon_0 c^3} \frac{1}{R^2} \left(\ddot{p} \right)^2 \sin^2\theta \, \vec{n}$$

$$P = \frac{1}{4\pi\varepsilon_0} \frac{1}{3c^3} \left(\ddot{p} \right)^2$$

5、短天线的辐射　辐射电阻

1）短天线辐射的偶极辐射近似

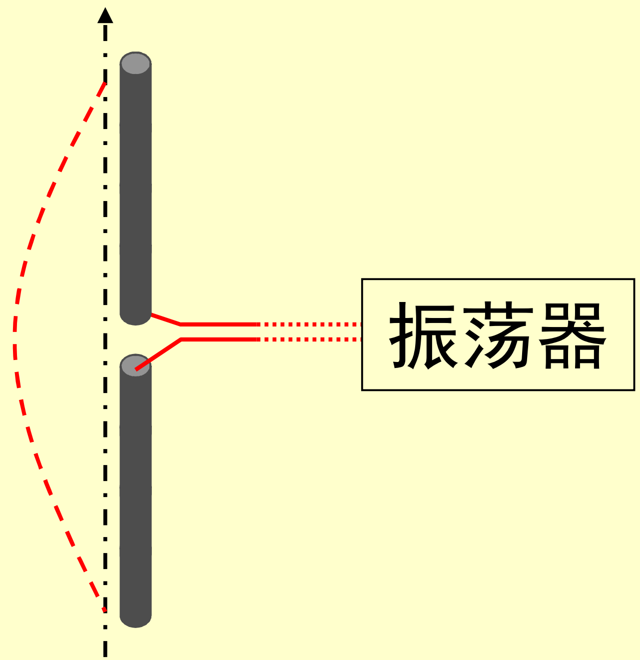
电偶极辐射近似：电荷体系的分布尺寸远小于辐射电磁波的波长

源点：$\vec{J}(\vec{x}',t)=\vec{J}(\vec{x}')e^{-i\omega t}$

2）短天线的辐射功率

对于中心馈电的直线短天线，

$$I_z = I_0\left(1-\frac{2}{\ell}|z|\right)$$



①馈电点处的电流最大；

②天线两端的电流为零。

$$\int \vec{J}(\vec{x}')\mathrm{d}V' = -\mathrm{i}\omega\vec{p}$$

$$\dot{\vec{p}} = -\mathrm{i}\omega\vec{p}\mathrm{e}^{-\mathrm{i}\omega t}$$

$$\left|\dot{\vec{p}}\right| = \int \vec{J}\left(\vec{x}'\right)\mathrm{d}V'$$

$$\left|\dot{\vec{p}}\right| = \int_{-\ell/2}^{\ell/2} I\left(z\right)\mathrm{d}z$$

$$= \int_{-\ell/2}^{\ell/2} I_0\left(1 - \frac{2}{\ell}\left|z\right|\right)\mathrm{d}z \quad = \frac{1}{2}I_0\ell$$

$$\ddot{\vec{p}} = -\mathrm{i}\omega\,\dot{\vec{p}}$$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{1}{3c^3}\left|\ddot{p}\right|^2$$

总辐射功率：

$$P = \frac{1}{4\pi\varepsilon_0}\frac{\omega^2}{3c^3}\left|\dot{\mathrm{p}}\right|^2 \quad = \frac{1}{4\pi\varepsilon_0}\frac{\omega^2}{3c^3}\frac{1}{4}I_0^2\ell^2 \quad = \frac{\omega^2}{48\pi\varepsilon_0 c^3}I_0^2\ell^2$$

$$= \frac{4\pi^2}{48\pi\varepsilon_0 c\left(cT\right)^2}I_0^2\ell^2 \quad = \frac{\pi}{12}\sqrt{\frac{\mu_0}{\varepsilon_0}}I_0^2\frac{\ell^2}{\lambda^2}$$

$$P=\frac{\pi}{12}\sqrt{\frac{\mu_0}{\varepsilon_0}}I_0^2\frac{\ell^2}{\lambda^2}$$

① 在$l<<\lambda$ 的情况下，对于给定的I_0，天线的辐射功率正比于 $(l/\lambda)^2$；

② 由于辐射功率正比于 I_0^2 ，如果把这种辐射等效成一个电阻上的损耗功率，则这个等效的电阻称为辐射电阻 R_r；

③ 能量不断的以电磁波的形式向外辐射，因此电源需要供给一定的功率来维持辐射。

$$P=\frac{\pi}{12}\sqrt{\frac{\mu_0}{\varepsilon_0}}I_0^2\frac{\ell^2}{\lambda^2}$$

3）短天线的辐射电阻：

$$P=\frac{1}{2}I_0^2R_r$$

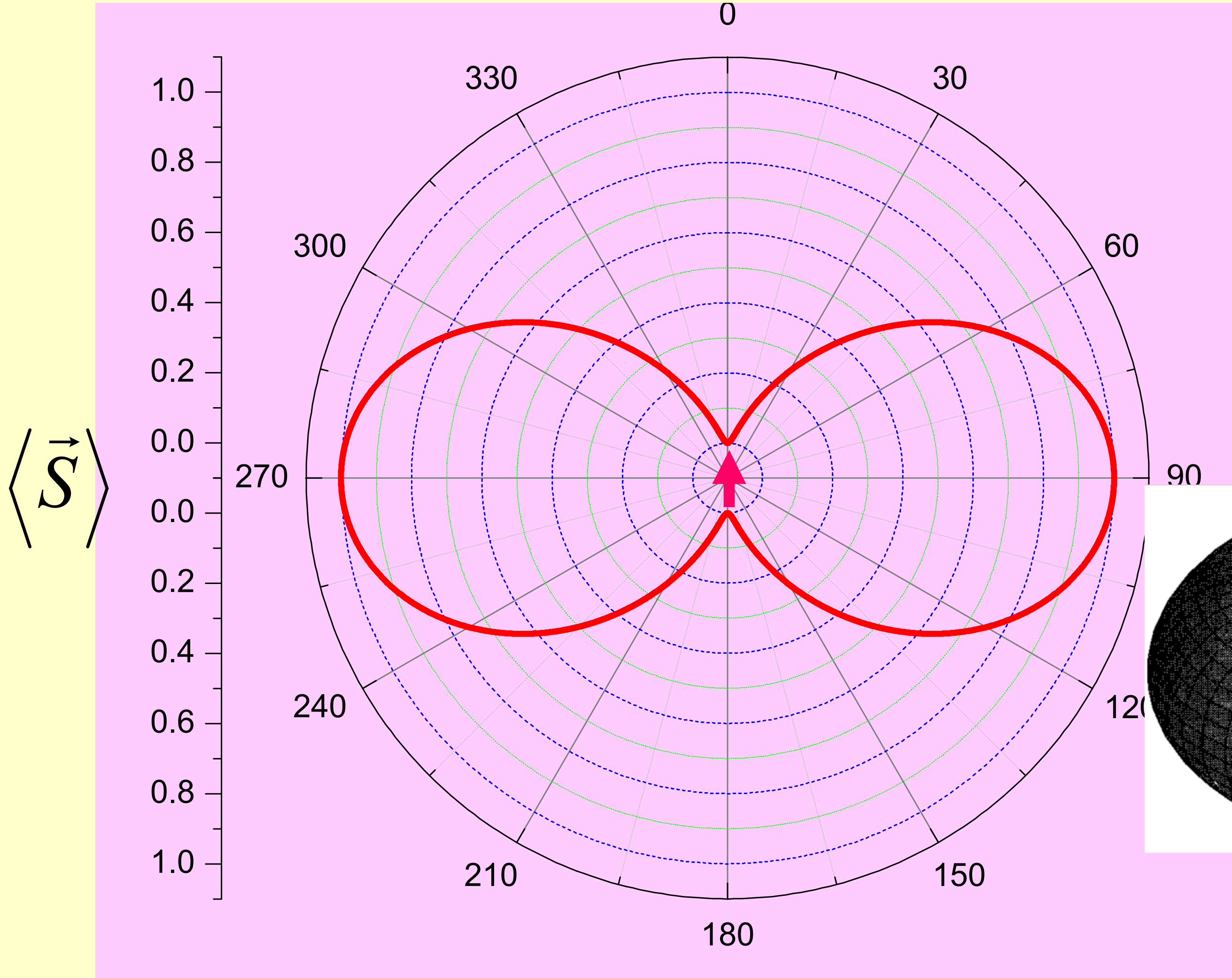
$$\sqrt{\frac{\mu_0}{\varepsilon_0}}=376.7\Omega$$

$$\Rightarrow\quad R_r=\frac{\pi}{6}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{\ell}{\lambda}\right)^2=187\left(\frac{\ell}{\lambda}\right)^2\Omega$$

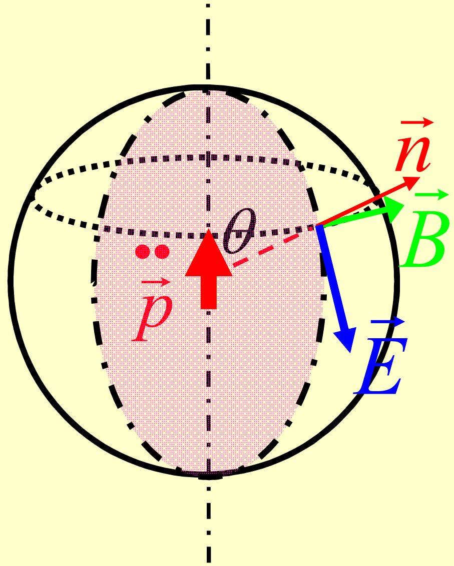
① 短天线的辐射能力有限；

② 要提高辐射功率，需将天线的长度增加到波长量级，但此时的辐射特性已经不能用偶极辐射来描述

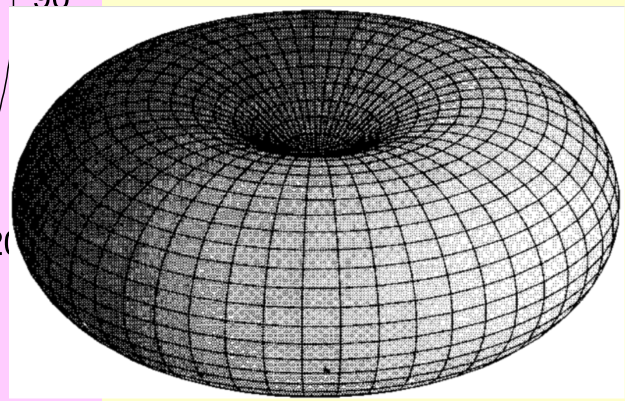
偶极子辐射



$\langle \vec{S} \rangle$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
0
30
60
90
120
150
180
210
240
270
300
330



$\vec{n}$
$\vec{B}$
θ
$\ddot{\vec{p}}$
$\vec{E}$

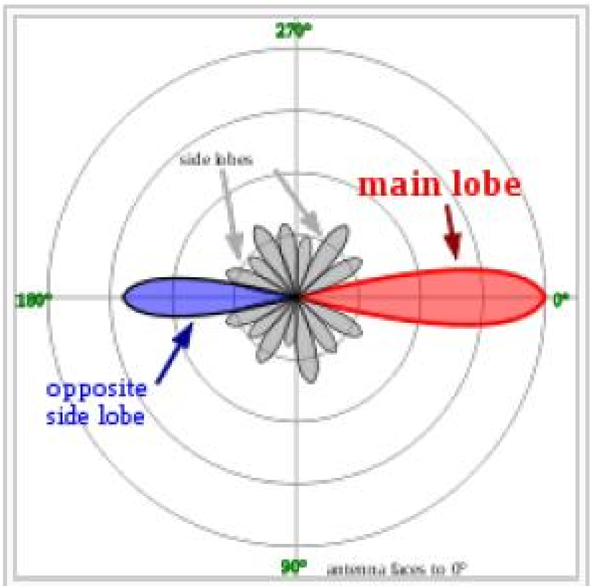


Antenna (radio):

An antenna (or aerial) is a transducer that transmits or receives electromagnetic waves.

Wikipedia

Yagi-Uda Antenna(八木宇田天线)



Yagi antenna used for mobile military communications station, Dresden, Germany, 1955

Wikipedia

“Rabbit ears” dipole antenna for television reception

Wikipedia

Cell phone base station antennas

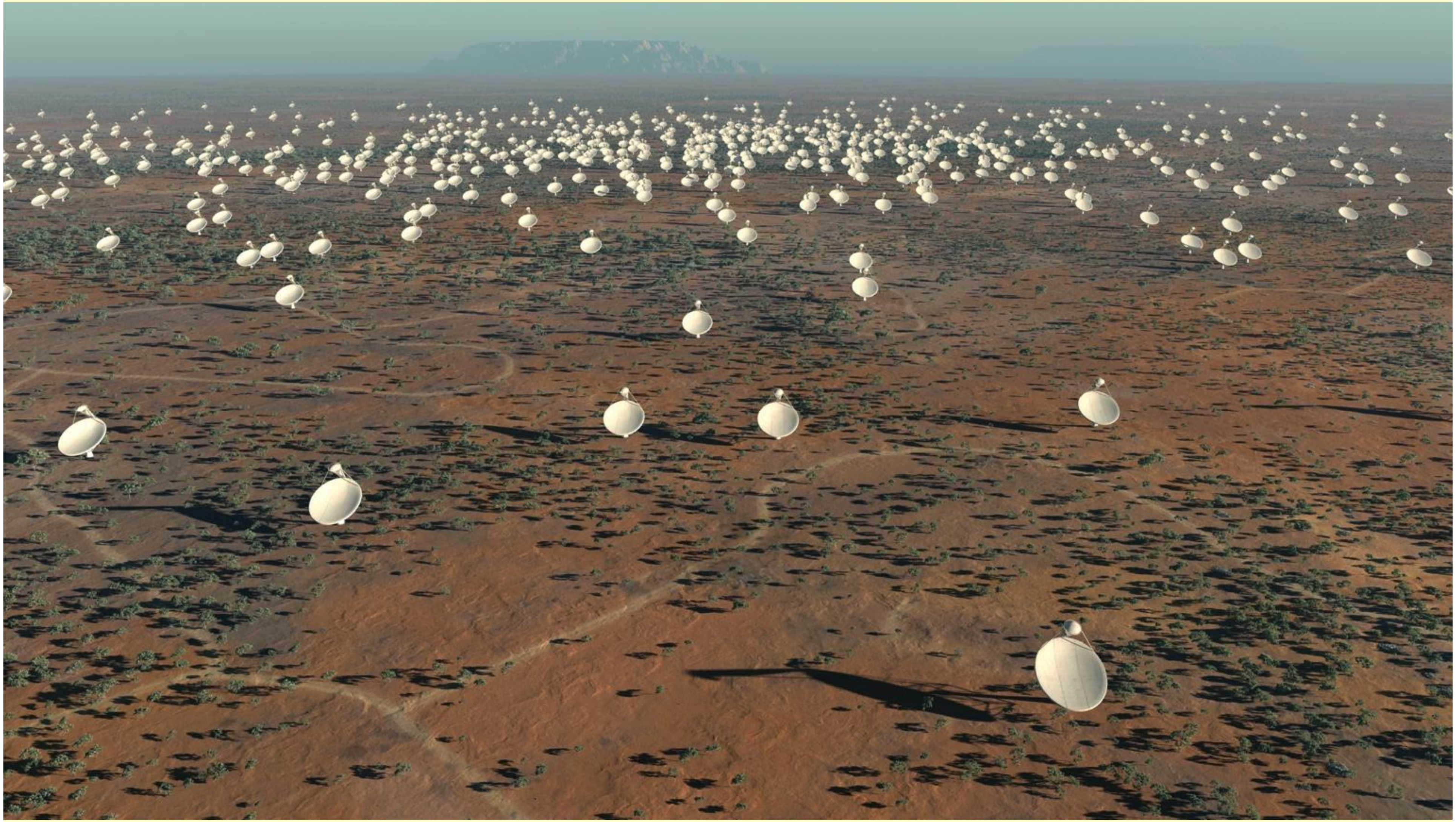
Cell phone base station antennas

Parabolic antenna

Parabolic antenna for communicating with spacecraft, Canberra, Australia

Wikipedia

天线阵列
(Antenna array)



Optical Antenna (光学天线)

- **Definition:** A device designed to efficiently convert free-propagating optical radiation to localized energy, and vice versa
- **Motivations:** Enable us to concentrate external radiation to dimensions smaller than diffraction limit

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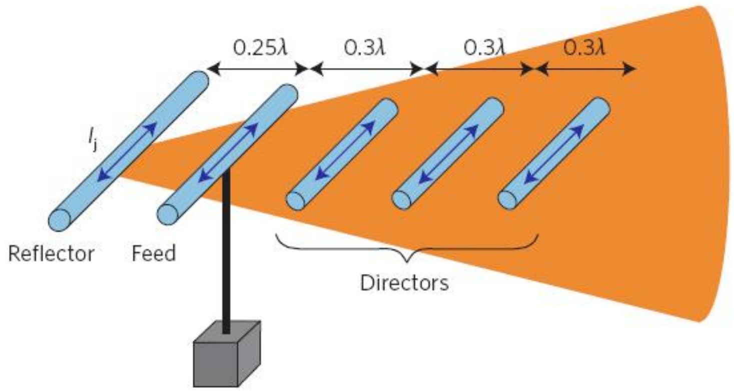
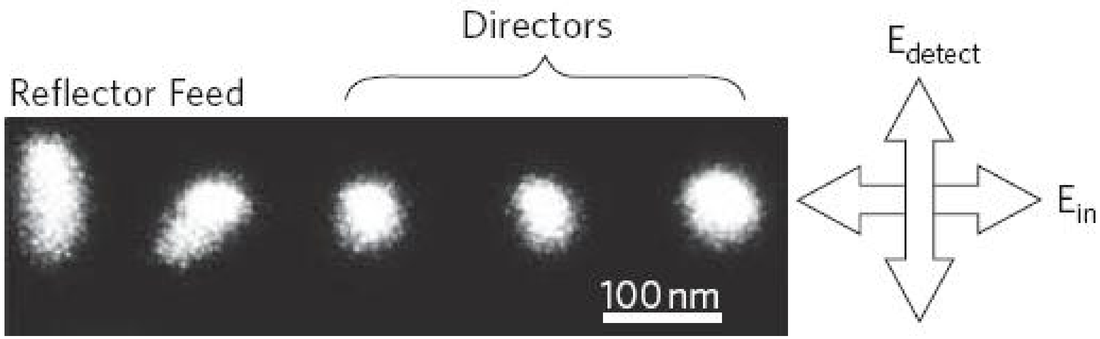


Figure 1 | Typical geometry of a five-element RF Yagi–Uda antenna.

Directional control of light by a nano-optical Yagi–Uda antenna

Terukazu Kosako, Yutaka Kadoya* and Holger F. Hofmann



See Nature Photonics 4,313 (2010)

Radiation patterns of five-element Yagi–Uda optical antenna